Quantum Field Theory Problems

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Question 1. *Consider the space $\mathcal{D}$ of continuously differentiable complex-valued functions f on [0,1]. Consider the operator A on $L^2([0,1])$ with domain $\mathcal{D}$, defined by $A(f) = if'$. Is A symmetric? What happens if one considers instead the domain $\mathcal{D}_\alpha := \{f \in \mathcal{D} : f(1) = \alpha f(0)\}$, where α is a complex number with modulus 1? Written by Prof. Sourav Chatterjee.*

Proof. We want to check if $\langle A\psi|\varphi\rangle = \langle\psi|A\varphi\rangle$. This gives us $\langle i\psi'|\varphi\rangle, \langle\psi|i\varphi'\rangle$. Rewriting our bra-kets into integrals, we have $\int_0^1 (i\psi')^*\varphi dx, \int_0^1 \psi^* i\varphi' dx$. Evaluating the former, we have $\int_0^1 (i\psi')^*\varphi dx = \int_0^1 (-i)\psi'^*\varphi dx = [-i\psi^*\varphi]_0^1 - \int_0^1 (-i)\psi^*\varphi' dx) \neq \int_0^1 i\psi^*\varphi' dx$. Thus, on this general a domain, A is not symmetric.

If instead our domain is D_α, then, evaluating the same integral, we have $\int_0^1 (i\psi')^*\varphi dx = [-i\psi^*\varphi]_0^1 - \int_0^1 (-i)\psi^*\varphi' dx = [-i\psi^*(1)\varphi(1) + i\psi^*(0)\varphi(0)] + \int_0^1 i\psi^*\varphi' dx$. Computing the first term, we have $[-i(\alpha\psi(0))^*\alpha\varphi(0) + i\psi^*(0)\varphi(0)] = [-i\alpha^*\alpha\psi^*(0)\varphi(0) + i\psi(0)\varphi(0)] = (1-\alpha^*\alpha)i\psi^*(0)\varphi(0)$. Since α has modulus 1, $\alpha^*\alpha = 1$, and this term becomes zero and hence $\int_0^1 (A\psi)^*\varphi dx = \int_0^1 \psi^* A\varphi$, so A becomes symmetric on this domain. □

Question 2. *Recall the definition of the manifold X_m, the measure λ_m on X_m, and the Hilbert space $\mathcal{H} = L^2(X_m, d\lambda_m)$. Recall also the operator valued distributions $a(p)$ and $a^\dagger(p)$ on the bosonic Fock space of $\mathcal{H}$. Finally, recall the definitions of $a(\mathbf{p})$ and $a^\dagger(\mathbf{p})$. Assuming the commutation relations for $a(p)$ and $a^\dagger(p)$ as given, prove that*

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\mathbb{1}$$

where $\mathbb{1}$ is the identity operator on the Fock space. Written by Prof. Sourav Chatterjee.

Proof. Integrating this operator in Schwartz space, we have $\int\int \frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{d^3\mathbf{p}}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p}')[a(\mathbf{p}), a^\dagger(\mathbf{p}')]$. Since $a(\mathbf{p}) = \frac{a(p)}{\sqrt{2w_\mathbf{p}}}, a^\dagger(\mathbf{p}') = \frac{a^\dagger(p')}{\sqrt{2w_{\mathbf{p}'}}}$, we can conclude $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = \frac{1}{\sqrt{4w_\mathbf{p}w_{\mathbf{p}'}}}[a(p), a^\dagger(p')]$. The first expression then becomes $\int\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{4w_\mathbf{p}w_{\mathbf{p}'}}} f(\mathbf{p})^* g(\mathbf{p}')[a(p), a^\dagger(p')]$. We know from the notes that $[a(p), a^\dagger(p')] = \delta(p-p')1$. We want to integrate this on our mass shell with respect to our probability measure in order to apply our useful distribution. Since $\int_{X_m} d\lambda_m(p) f(p) = \int_{\mathbb{R}^3}\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2w_\mathbf{p}} f(w_\mathbf{p}, \mathbf{p})$, we have the equality

$$\int\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{1}{\sqrt{4w_{\mathbf{p}}w_{\mathbf{p}'}}} f(\mathbf{p})^* g(\mathbf{p}')[a(p), a^\dagger(p')] =$$
$$\int\int d\lambda_m(p) d\lambda_m(p') \sqrt{4w_{\mathbf{p}}w_{\mathbf{p}'}} f(\mathbf{p})^* g(\mathbf{p}')[a(p), a^\dagger(p')]$$

Integrating once, we find this is equal to $\int d\lambda_m(p)\sqrt{4w_{\mathbf{p}}^2} f(\mathbf{p})^* g(\mathbf{p})1 = \int d\lambda_m(p) 2w_{\mathbf{p}} f(\mathbf{p})^* g(\mathbf{p})1$. Going back to integrating over momentum space, we find that this is equal to $\int \frac{d^3\mathbf{p}}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p})1$, where 1 is the identity operator on our Fock space.

Now we consider $\int\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}'}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p}')(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')1$. Integrating once, we find this gives us $\int \frac{d^3\mathbf{p}}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p})1$, the exact result (up to a set of measure zero) as our original commutator. Thus, $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')$. □

Question 3. *Consider the theory for massive scalar bosons of mass m. Let φ be the free field of this theory, and let H_0 be the Hamiltonian for free evolution. Give a formal proof of the relation*

$$\frac{\partial \varphi}{\partial t} = i[H_0, \varphi]$$

Written by Prof. Sourav Chatterjee.

Proof. Suppose we have a Schwartz function f. Then, since $H_0 = \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3} w_{\mathbf{p}} a^\dagger(\mathbf{p}) a(\mathbf{p})$ and $\varphi(f) = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{1}{\sqrt{2w_{\mathbf{p}'}}}(e^{-i(x,p)}a(\mathbf{p}') + e^{i(x,p)}a^\dagger(\mathbf{p}'))$, we have

$$(H_0\varphi)(f) = \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3} w_{\mathbf{p}} a^\dagger(\mathbf{p}) a(\mathbf{p}) \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{1}{\sqrt{2w_{\mathbf{p}'}}}(e^{-i(x,p)}a(\mathbf{p}' + e^{i(x,p)}a^\dagger(\mathbf{p}'),$$
$$(\varphi H_0)(f) = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{1}{\sqrt{2w_{\mathbf{p}'}}}(e^{-i(x,p)}a(\mathbf{p}' + e^{i(x,p)}a^\dagger(\mathbf{p}') \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3} w_{\mathbf{p}} a^\dagger(\mathbf{p}) a(\mathbf{p})$$

Thus we have

$$[H_0, \varphi](f) = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{w_{\mathbf{p}}}{\sqrt{2w_{\mathbf{p}'}}} A, \text{ where}$$
$$A =$$
$$a^\dagger(\mathbf{p})a(\mathbf{p})e^{-i(x,p)}a(\mathbf{p}') + a^\dagger(\mathbf{p})a(\mathbf{p})e^{i(x,p')}a^\dagger(\mathbf{p}') - e^{-i(x,p')}a(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p}) - e^{i(x,p')}a^\dagger(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p})$$

Factoring out scalars, we have

$$A = e^{-i(x,p')}(a^\dagger(\mathbf{p})a(\mathbf{p})a(\mathbf{p}')) - a(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p}) + e^{i(x,p')}(a^\dagger(\mathbf{p})a(\mathbf{p})a^\dagger(\mathbf{p}') - a^\dagger(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p}))$$

Because $[a(\mathbf{p}), a(\mathbf{p}')] = 0$ and $[a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')] = 0$, this is equal to

$$e^{-i(x,p')}(a^\dagger(\mathbf{p})a(\mathbf{p}')a(\mathbf{p}) - a(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p})) + e^{i(x,p')}(a^\dagger(\mathbf{p})a(\mathbf{p})a^\dagger(\mathbf{p}') - a^\dagger(\mathbf{p})a^\dagger(\mathbf{p}')a(\mathbf{p}))$$
$$= e^{-i(x,p')}[a^\dagger(\mathbf{p}), a(\mathbf{p}')]a(\mathbf{p}) + e^{i(x,p')}a^\dagger(\mathbf{p})[a(\mathbf{p}), a^\dagger(\mathbf{p}')]$$

We know from the previous problem that $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')$. Also, notice that $[A,B] = AB - BA = (-1)(BA - AB) = -[B,A]$. Thus, A becomes

$$e^{-i(x,p')}(-1)(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')a(\mathbf{p}) + e^{i(x,p')}a^\dagger(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')$$
$$= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')(e^{i(x,p')}a^\dagger(\mathbf{p}) - e^{-i(x,p')}a(\mathbf{p}))$$

Now, with this helpful rearrangement, we have $[H_0,\varphi](f) =$

$$\int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{w_{\mathbf{p}}}{\sqrt{2w_{\mathbf{p}'}}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')(e^{i(x,p')}a^\dagger(\mathbf{p}) - e^{-i(x,p')}a(\mathbf{p}))$$
$$= \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{w_{\mathbf{p}}}{\sqrt{2w_{\mathbf{p}}}}(e^{i(x,p)}a^\dagger(\mathbf{p}) - e^{-i(x,p)}a(\mathbf{p}))$$

Let's take the time derivative of $\varphi(f)$ and see what we get. Notice that $(x,p) = tw_{\mathbf{p}} + \mathbf{x}\cdot\mathbf{p}$, so the time derivative of $e^{\pm i(x,p)} = \pm i w_{\mathbf{p}} e^{\pm i(x,p)}$. Thus, $\frac{\partial\varphi}{\partial t} = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{iw_{\mathbf{p}'}}{\sqrt{2w_{\mathbf{p}'}}}(-e^{-i(x,p)}a(\mathbf{p}')+ e^{i(x,p)}a^\dagger(\mathbf{p}'))$. This is simply i times the previous expression we derived form the commutator. Thus, $\frac{\partial\varphi}{\partial t} = i[H_0,\varphi]$, up to a set of measure zero. □

Question 4. *In φ^4 field theory, compute the first order term in the perturbative expansion of the scattering amplitude*

$$\langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 | S | \mathbf{p}_1\rangle$$

Written by Prof. Sourav Chatterjee.

Proof. In a first order Dyson series expansion of S gives us $1 - \frac{ig}{4!}\int_{\mathbb{R}} d^4x : \varphi(x)^4 : + \mathcal{O}(g^2)$. We then have

$$\langle \mathbf{p_2},\mathbf{p_3},\mathbf{p_4}|S|\mathbf{p_1}\rangle = \langle \mathbf{p_2},\mathbf{p_3},\mathbf{p_4}|\mathbf{p_1}\rangle - \frac{ig}{4!}\int_{\mathbb{R}} d^4x \langle \mathbf{p_2},\mathbf{p_3},\mathbf{p_4}| : \varphi(x)^4 : |\mathbf{p_1}\rangle + \mathcal{O}(g^2)$$
$$= \langle \mathbf{p_2},\mathbf{p_3},\mathbf{p_4}|\mathbf{p_1}\rangle - \frac{ig}{4!}\int_{\mathbb{R}} d^4x \langle 0|a(\mathbf{p_2})a(\mathbf{p_3})a(\mathbf{p_4}) : \varphi(x)^4 : a^\dagger(\mathbf{p_1})|0\rangle + \mathcal{O}(g^2)$$

For the first term, we notice that $\langle \mathbf{p_2},\mathbf{p_3},\mathbf{p_4}|\mathbf{p_1}\rangle = \langle 0|a(\mathbf{p_2})a(\mathbf{p_3})a(\mathbf{p_4})a^\dagger(\mathbf{p_1})|0\rangle$. Applying the first two operators we get either ground state back if $\mathbf{p_1} = \mathbf{p_4}$ or 0 if not. Annihilating the ground state with the third operator, we get 0, so in both cases $\langle \mathbf{p_2},\mathbf{p_3},\mathbf{p_4}|\mathbf{p_1}\rangle = 0$. Focusing on the integrand, we recall the following useful rules: $\langle 0|a(\mathbf{p})\varphi(x)|0\rangle = \frac{e^{i(x,p)}}{\sqrt{2w_{\mathbf{p}}}}$, $\langle 0|\varphi(x)a^\dagger(\mathbf{p})|0\rangle = \frac{e^{-i(x,p)}}{\sqrt{2w_{\mathbf{p}}}}$. $\langle 0|a(\mathbf{p_2})a(\mathbf{p_3})a(\mathbf{p_4}) : \varphi(x)^4 : a^\dagger(\mathbf{p_1})|0\rangle = \langle 0|a(\mathbf{p_2})\varphi(x)|0\rangle\langle 0|a(\mathbf{p_3})\varphi(x)|0\rangle\langle 0|a(\mathbf{p_4})\varphi(x)|0\rangle\langle 0|a^\dagger(\mathbf{p_1})\varphi(x)|0\rangle$. This expression is equal to $(e^{i(x,p_2+p_3+p_4-p_1)})/(\sqrt{16w_{\mathbf{p_2}}w_{\mathbf{p_3}}w_{\mathbf{p_4}}w_{\mathbf{p_1}}})$ for each suitable contraction diagram. Since the scattering involves 1 incoming particle and three outgoing particles, we want to consider all contraction diagrams of the "four all connected to the center $\varphi(x)$ operator"-shape. The $\varphi(x)^4$ operator has 4 tails, to which the incoming and outgoing particles get connected. Since there are 8 operators, there are $(8-1)!!$ diagrams, and 4! diagrams of this type. Thus we have 4! $(e^{i(x,p_2+p_3+p_4-p_1)})/(\sqrt{16w_{\mathbf{p_2}}w_{\mathbf{p_3}}w_{\mathbf{p_4}}w_{\mathbf{p_1}}})$ terms. Sticking these back into our integral and integrating, we get $(-\frac{ig}{4!}(4!)(2\pi)^4\delta^{(4)}(p_2+p_3+p_4-p_1))/(\sqrt{16w_{\mathbf{p_2}}w_{\mathbf{p_3}}w_{\mathbf{p_4}}w_{\mathbf{p_1}}})$. Thus we have $\langle \mathbf{p_2},\mathbf{p_3},\mathbf{p_4}|S|\mathbf{p_1}\rangle = (-ig(2\pi)^4\delta^{(4)}(p_2+p_3+p_4-p_1))/(\sqrt{16w_{\mathbf{p_2}}w_{\mathbf{p_3}}w_{\mathbf{p_4}}w_{\mathbf{p_1}}}) + \mathcal{O}(g^2)$. □

Question 5. 1. *Derive Maxwell's equations as the Euler-Lagrange equations of the action*

$$S = \int d^4x(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}),\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

treating the components $A_\mu(x)$ as the dynamical variables. Write the equations in standard from by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$. Construct the energy-momentum tensor for this theory.

2. *Construct the energy-momentum tensor for this theory. Note that the usual procedure does not result in a symmetric tensor. To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so*

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu,$$

leads to an energy-momentum tensor $\hat{T}$ that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \frac{1}{2}(E^2 + B^2); S = E \times B$$

Peskin & Schroeder 2.1.

Proof. 1. Let's first calculate $F^{\mu\nu}$. Given our identification with E^i and B^i, we have

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

Treating A_ν as our dynamical variables, we take

$$\begin{aligned} \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial \mathcal{L}}{\partial A_\nu} &= 0 \\ \partial_\mu \frac{\partial}{\partial(\partial_\mu A_\nu)}[-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)] &= 0 \\ \partial_\mu \frac{\partial}{\partial(\partial_\mu A_\nu)}[-\frac{1}{4}(2\partial_\mu A_\nu \partial_\mu A_\nu - 2\partial_\nu A_\mu \partial_\mu A_\nu)] &= 0 \\ \partial_\mu[-\frac{1}{4}(4\partial_\mu A_\nu - 4\partial_\nu A_\mu)] &= 0 \\ \partial_\mu F^{\mu\nu} &= 0 \end{aligned}$$

With the identification $F^{0i} = -E^i, F^{ij} = -\epsilon^{ijk}B^k$, we have $-\frac{\partial E}{\partial t} - \partial_i \epsilon^{ijk} B^k = 0$. Because I always forget the Levi-Civita symbols, we recall that

$$\epsilon^{ijk}\partial_j v_k = (\nabla \times v)^i$$

and thus $-\frac{\partial E}{\partial t} + \epsilon^{jik}\partial_i B^k = 0$, or

$$\nabla \times B = \frac{\partial E}{\partial t}$$

2. With this construction, we have

$$\begin{aligned}
\hat{T}^{\mu\nu} &= T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \\
&= \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\gamma)} \partial^\nu A_\gamma - \mathcal{L}\delta^{\mu\nu} + \partial_\lambda F^{\mu\lambda} A^\nu \\
&= -F^{\mu\gamma}\partial^\nu A_\gamma + \frac{1}{4} F_{\mu\nu}F^{\mu\nu} g^{\mu\nu} + \partial_\lambda (F^{\mu\lambda} A^\nu) \\
&= F^{\mu\iota}(\partial_\iota A^\nu - \partial^\nu A_\iota) + \frac{1}{4} F_{\mu\nu}F^{\mu\nu} g^{\mu\nu} - \partial_\lambda F^{\lambda\mu} A^\nu \\
&= F^{\mu\iota} F^\nu_\iota + \frac{1}{4} F_{\mu\nu}F^{\mu\nu} g^{\mu\nu} - (0) A^\nu
\end{aligned}$$

This is now a viable energy-momentum tensor. We now $\hat{T^{00}}$ and $\hat{T^{0i}}$:

$$\begin{aligned}
\hat{T^{00}} &= F^{0\iota} F^0_\iota + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\
&= E^\iota E_\iota + \frac{1}{4} F_{\mu\nu} F^{\mu\nu}
\end{aligned}$$

We then have

$$\begin{aligned}
\langle , \rangle = tr(\overline{F^{\mu\nu}} F^{\mu\nu}) &= \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_Z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_Z & -B_y & B_x & 0 \end{pmatrix} \\
&= tr(\begin{pmatrix} -E^2 & & & \\ & -E_x^2 + B_z^2 + B_y^2 & & \\ & & -E_y^2 + B_z^2 + B_x^2 & \\ & & & -E_z^2 + B_x^2 + B_y^2 \end{pmatrix}
\end{aligned}$$

This is equal to $2(B^2 - E^2)$. Thus we have that $\hat{T^{\mu\nu}} = E^2 + \frac{1}{4} 2(B^2 - E^2) = \frac{1}{2}(E^2 + B^2)$.

For $\hat{T^{0i}}$, we have

$$\begin{aligned}
\hat{T^{0i}} &= F^{0j} F^i_j + \frac{1}{2}(B^2 - E^2) g^{0i} \\
&= E^j \epsilon_{jik} B^k g^{mi} + \frac{1}{2}(B^2 - E^2) g^{0i} \\
&= \mathbf{E} \times \mathbf{B} = \mathbf{S}
\end{aligned}$$

□

Question 6. *Consider the the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is*

$$S = \int d^4x(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi)$$

(a) *Find the conjugate momenta to $\phi(x), \phi^*(x)$ and the canonical commutation relations. Show that the Hamiltonian is*

$$H = \int d^3x(\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi)$$

Compute the Heisenberg equation of motion for $\phi(x)$ and show that it is indeed the Klein-Gordon equation.

(b) *Diagonlize H by introducing creation and annihilation operators. Show that the theory contains two sets of particles of mass m.*

(c) *Rewrite the conserved charge*

$$Q = \int d^3x \frac{i}{2}(\phi^* \pi^* - \pi \phi)$$

in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

(d) *Consider the case of two complex Klein-Gordon fields with the same mass. Label the fields as $\phi_a(x)$, where a = 1,2. Show that there are now four conserved charges, one given by the generalization of part (c), and the other three given by*

$$Q^i = \int d^3x \frac{i}{2}(\phi_a^* (\sigma^i)_{ab} \pi_b^* - \pi_a (\sigma^i)_{ab} \phi_b)$$

where σ^i are the Pauli sigma matrices. Show that these three charges have the commutation relations of angular momentum ($SU(2)$). Generalize these results to the case of n identical complex scalar fields.

Peskin & Schroeder, 2.2.

Proof. (a) We have that $p(\mathbf{x}) = \frac{\partial L}{\partial \dot{\phi}(\mathbf{x})} = \frac{\partial}{\partial \dot{\phi}(\mathbf{x})} \int d^4x (\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi) = \frac{\partial}{\partial \dot{\phi}(\mathbf{x})} \int d^4x (\partial_\mu \phi^* g^\mu_\nu \partial^\nu \phi - m^2 \phi^* \phi) = \frac{\partial}{\partial \dot{\phi}(\mathbf{x})} \int d^4x (\frac{\partial \phi^*}{\partial t} \frac{\partial \phi}{\partial t} - \nabla \phi^* \cdot \nabla \phi - m^2 \phi^* \phi) = \frac{\partial}{\partial \dot{\phi}(\mathbf{x})} \int d^4x (\dot{\phi}^* \dot{\phi} - \nabla \phi^* \cdot \nabla \phi - m^2 \phi^* \phi)$. Thus, $\pi = \dot{\phi}^*$. Similarly, $\pi^* = \dot{\phi}$. Since ϕ, ϕ^* are the dynamical variables, the canonical commutation relations are

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = [\phi^*(\mathbf{x}), \pi^*(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}),$$
$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = [\phi^*(\mathbf{x}), \phi^*(\mathbf{y})] = [\pi(\mathbf{x}), \pi(\mathbf{y})] = [\pi^*(\mathbf{x}), \pi^*(\mathbf{y})] = 0$$

from quantization of the Klein-Gordon field given in the textbook. Given the equation for the Hamiltonian, we have

$$\begin{aligned} H &= \int d^3x [\sum_{a,b} \pi_i(\mathbf{x}) \dot{\phi}_i(\mathbf{x}) - \mathcal{L}] \\ &= \int d^3x [\pi^* \dot{\phi}^* + \pi \dot{\phi} - \mathcal{L}] \\ &= \int d^3x [\dot{\phi} \dot{\phi}^* + \dot{\phi}^* \dot{\phi} - \mathcal{L}] \\ &= \int d^3x [2 \dot{\phi} \dot{\phi}^* - \dot{\phi} \dot{\phi}^* + \nabla \phi \cdot \nabla \phi^* + m^2 \phi^* \phi] \\ &= \int d^3x [\dot{\phi} \dot{\phi}^* + \nabla \phi \cdot \nabla \phi^* + m^2 \phi^* \phi] \\ &= \int d^3x [\pi^* \pi + \nabla \phi \cdot \nabla \phi^* + m^2 \phi^* \phi] \end{aligned}$$

We want to compute $i\frac{\partial\phi}{\partial t}$ via the Heisenberg Equation of Motion, so we calculate $[\phi, H]$.

$$
\begin{aligned}
i\frac{\partial\phi}{\partial t} &= [\phi, H] \\
&= [\phi(x'), \int d^3x(\pi^*\pi + \nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi)] \\
&= \int d^3x[\phi(x'), \pi^*\pi + \nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi] \\
&= \int d^3x([\phi(x'), \pi^*\pi] + [\phi, \nabla\phi^*\cdot\nabla\phi] + m^2[\phi, \phi^*\phi]) \\
&= \int d^3x\delta^{(3)}(x'-x)i\pi^*(x) \\
&= i\pi^*(x) \\
i\frac{\partial\pi^*}{\partial t} &= [\pi^*, H] \\
&= [\pi^*(x'), \int d^3x(\pi^*\pi + \nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi)] \\
&= \int d^3x[\pi^*(x'), \pi^*\pi + \nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi] \\
\text{(integrating by parts)} \quad &= \int d^3x([\pi^*(x'), \pi^*\pi] + [\pi^*(x'), \phi^*(-\nabla^2 + m^2)\phi]) \\
&= \int d^3x\delta^{(3)}(x'-x)(-i)(-\nabla^2 + m^2)\phi(x) \\
&= i(\nabla^2 - m^2)\phi
\end{aligned}
$$

Since $i\frac{\partial\phi}{\partial t} = i\pi^*$ and $i\frac{\partial\pi^*}{\partial t} = i(\nabla^2 - m^2)\phi$, so $\frac{\partial^2\phi}{\partial t^2} = (\nabla^2 - m^2)\phi$, which is the Klein-Gordon equation.

(b) Since ϕ satisfies the Klein-Gordon equation, and, in the same way, so does ϕ^*, we take the Fourier transform to gain more insight into $\nabla^2\phi$:

$$
\begin{aligned}
\phi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}}\phi(\mathbf{p}) \Rightarrow \\
[\frac{\partial^2}{\partial t^2} + (p^2 + m^2)]\phi(\mathbf{p}) = 0,\ [\frac{\partial^2}{\partial t^2} + (p^2 + m^2)]\phi^*(\mathbf{p}) = 0
\end{aligned}
$$

We write ϕ in terms of two real valued scalar free fields ψ_1, ψ_2, of which we already know the theory:

$$
\phi = \frac{\psi_1 + i\psi_2}{\sqrt{2}},\ \phi^* = \frac{\psi_1 - i\psi_2}{\sqrt{2}}
$$

Since ψ_1, ψ_2 are independent free fields, both must satisfy the harmonic oscillator equation:

$$\tfrac{1}{\sqrt{2}}[\tfrac{\partial^2}{\partial t^2} + (p^2 + m^2)]\psi_1 = 0,\ \tfrac{\pm i}{\sqrt{2}}[\tfrac{\partial^2}{\partial t^2} + (p^2 + m^2)]\psi_2 = 0 \Rightarrow$$
$$[\tfrac{\partial^2}{\partial t^2} + (p^2 + m^2)]\psi_1 = 0,\ [\tfrac{\partial^2}{\partial t^2} + (p^2 + m^2)]\psi_2 = 0 \Rightarrow$$
$$\omega_1 = \sqrt{p_1^2 + m^2},\ \omega_2 = \sqrt{p_2^2 + m^2}$$

Since the frequencies of the oscillators have independent momentums and ϕ is not hermitian, we create two different creation and annihilation operators:

$$a_i = \sqrt{\frac{\omega_i}{2}}q_i + \frac{i}{\sqrt{2\omega_i}}p_i,\ a_i^\dagger = \sqrt{\frac{\omega_i}{2}}q_i - \frac{i}{\sqrt{2\omega_i}}p_i,\ i \in \{1, 2\}$$

where $q_1 = \phi, q_2 = \phi^*, p_1 = \pi, p_2 = \pi^*$, with the notation in the spirit of Peskin and Schroeder. These creation operators, given their frequencies, represent creating two different particles with mass m. From the theory of a real-valued scalar free field, we know that

$$\phi = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}}(a_1(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} + a_2^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}})$$
$$\phi^* = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}}(a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} + a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}})$$

The two different operators ensure that ϕ is not hermitian. From above we know that $\pi = \dot{\phi}^*, \pi^* = \dot{\phi}$, and, using our real-valued scalar free field as reference, we have

$$\pi = \int \frac{d^3\mathbf{p}}{(2\pi)^3} i\sqrt{\frac{\omega_\mathbf{p}}{2}}(a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} - a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}})$$
$$\pi^* = \int \frac{d^3\mathbf{p}}{(2\pi)^3} (-i)\sqrt{\frac{\omega_\mathbf{p}}{2}}(a_1(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} - a_2^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}})$$

Finally, we rewrite our Hamiltonian in terms of our operators:

$$
\begin{aligned}
H &= \int d^3\mathbf{x}(\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi) \\
&= \int d^3\mathbf{x}(\int\int \frac{d^3\mathbf{p}d^3\mathbf{p}'}{(2\pi)^6}\frac{\sqrt{\omega_\mathbf{p}\omega_{\mathbf{p}'}}}{2}\{a_1(\mathbf{p})a_1^\dagger(\mathbf{p}')e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} - a_1(\mathbf{p})a_2(\mathbf{p}')e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} \\
&\quad - a_2^\dagger(\mathbf{p})a_1^\dagger(\mathbf{p}')e^{-i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} + a_2^\dagger(\mathbf{p})a_2(\mathbf{p}')e^{i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}}\} \\
&+ \int\int \frac{d^3\mathbf{p}d^3\mathbf{p}'}{(2\pi)^6}\frac{1}{2\sqrt{\omega_\mathbf{p}\omega_{\mathbf{p}'}}}[-i\mathbf{p}a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} + i\mathbf{p}a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}] \\
&\quad \cdot [i\mathbf{p}'a_1(\mathbf{p}')e^{i\mathbf{p}'\cdot\mathbf{x}} - i\mathbf{p}'a_2^\dagger(\mathbf{p}')e^{-i\mathbf{p}'\cdot\mathbf{x}}] \\
&+ m^2\int\int \frac{d^3\mathbf{p}d^3\mathbf{p}'}{(2\pi)^6}\frac{1}{2\sqrt{\omega_\mathbf{p}\omega_{\mathbf{p}'}}}\{a_1^\dagger(\mathbf{p})a_1(\mathbf{p}')e^{i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}} + a_1^\dagger(\mathbf{p})a_2^\dagger(\mathbf{p}')e^{-i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} \\
&\quad + a_2(\mathbf{p})a_1(\mathbf{p}')e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} + a_2(\mathbf{p})a_2^\dagger(\mathbf{p}')e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\}) \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3}(\frac{\omega_\mathbf{p}}{2}\{a_1(\mathbf{p})a_1^\dagger(\mathbf{p}) - a_1(\mathbf{p})a_2(-\mathbf{p}) - a_2^\dagger(\mathbf{p})a_1^\dagger(-\mathbf{p}) + a_2^\dagger(\mathbf{p})a_2(\mathbf{p})\} \\
&+ \frac{p^2}{2\omega_\mathbf{p}}\{a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) + a_1^\dagger(\mathbf{p})a_2^\dagger(-\mathbf{p}) + a_2(\mathbf{p})a_1(-\mathbf{p}) + a_2(\mathbf{p})a_2^\dagger(\mathbf{p})\} \\
&+ \frac{m^2}{2\omega_\mathbf{p}}\{a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) + a_1^\dagger(\mathbf{p})a_2^\dagger(-\mathbf{p}) + a_2(\mathbf{p})a_1(-\mathbf{p}) + a_2(\mathbf{p})a_2^\dagger(\mathbf{p})\}) \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3}[\frac{\omega_\mathbf{p}}{2}\{a_1(\mathbf{p})a_1^\dagger(\mathbf{p}) - a_1(\mathbf{p})a_2(-\mathbf{p}) - a_2^\dagger(\mathbf{p})a_1^\dagger(-\mathbf{p}) + a_2^\dagger(\mathbf{p})a_2(\mathbf{p})\} \\
&+ \frac{p^2+m^2}{2\omega_\mathbf{p}}\{a_1(\mathbf{p})a_1^\dagger(\mathbf{p}) + a_1^\dagger(\mathbf{p})a_2^\dagger(-\mathbf{p}) + a_2(\mathbf{p})a_1(-\mathbf{p}) + a_2^\dagger(\mathbf{p})a_2(\mathbf{p})\}]
\end{aligned}
$$

where in (70) the middle two terms have a positive sign because we subtract $\mathbf{p}\cdot\mathbf{p}' = \mathbf{p}\cdot(-\mathbf{p})$. Furthermore, since $\mathbf{p} \neq -\mathbf{p}$, we can commute our operators. Thus we have the Hamiltonian as

$$H = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\omega_\mathbf{p}\{a_1a_1^\dagger + a_2a_2^\dagger\}$$

Since this Hamiltonian is constructed purely out of constants and operators whose eigenvectors are momentum eigenstates, our Hamiltonian is now diagonalized. The indices 1 and 2 represent the two particles of mass m.

(c) This is just plugging in our values for momentum and position and integrating, like the previous problem. We have

$$
\begin{aligned}
Q &= \int d^3\mathbf{x}\frac{i}{2}(\phi^*\pi^* - \pi\phi) \\
&= \int d^3\mathbf{x}\frac{i}{2}\int d^3\mathbf{p}\int d^3\mathbf{p}'\frac{1}{(2\pi)^6}\frac{1}{2}[(-i)(a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} + a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}})(a_1(\mathbf{p}')e^{i\mathbf{p}'\cdot\mathbf{x}} - a_2^\dagger(\mathbf{p}')e^{-i\mathbf{p}'\cdot\mathbf{x}}) \\
&- i(a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} - a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}})(a_1(\mathbf{p}')e^{i\mathbf{p}'\cdot\mathbf{x}} + a_2^\dagger(\mathbf{p}')e^{-i\mathbf{p}'\cdot\mathbf{x}})] \\
&= \int d^3\mathbf{p}\frac{1}{(2\pi)^3}\frac{1}{4}([a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) - a_1^\dagger(\mathbf{p})a_2^\dagger(-\mathbf{p}) + a_2(\mathbf{p})a_1(-\mathbf{p}) - a_2(\mathbf{p})a_2^\dagger(\mathbf{p})] \\
&+ [a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) + a_1^\dagger(\mathbf{p})a_2^\dagger(-\mathbf{p}) - a_2(\mathbf{p})a_1(-\mathbf{p}) - a_2(\mathbf{p})a_2^\dagger(\mathbf{p})]) \\
&= \frac{1}{2}\int d^3\mathbf{p}\frac{1}{(2\pi)^3}[a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) - a_2(\mathbf{p})a_2^\dagger(\mathbf{p})]
\end{aligned}
$$

This means that this theory has two particle types: one created by $a_1^\dagger(\mathbf{p})$ and one created by $a_2^\dagger(\mathbf{p})$. In examining $[Q, a_i^\dagger]\,|n\rangle$ for some state n-particle state $|n\rangle$, we can deduce the charge. It is easy to see that $[a_1, a_2] = 0, [a_i, a_i^\dagger] = 1$ since ψ_1, ψ_2 are independent fields. Thus

$$
[Q, a_1^\dagger] = a_1^\dagger, [Q, a_2^\dagger] = -a_2^\dagger
$$

This means that the charges are valued at 1 unit for particles created by $a_1^\dagger$ and -1 for particles created by $a_2^\dagger$].

(d) For two complex scalar fields, the lagrangian is then

$$
\mathcal{L} = \partial_\mu\phi_1^*\partial^\mu\phi_1 - m^2\phi_1^*\phi_1 + \partial_\mu\phi_2^*\partial^\mu\phi_2 - m^2\phi_2^*\phi_2
$$

We then have

$$
\begin{aligned}
j^\mu(x) &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\Delta\phi_1 + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)}\Delta\phi_2 + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1^*)}\Delta\phi_1^* + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2^*)}\Delta\phi_2^* - \mathcal{J}^\mu \\
&= \partial^\mu\phi_1^*\Delta\phi_1 + \partial^\mu\phi_2^*\Delta\phi_2 + \partial^\mu\phi_1\Delta\phi_1^* + \partial^\mu\phi_2\Delta\phi_2^* - \mathcal{J}^\mu \\
\to Q &= \int d^3x(\pi_1^*\Delta\phi_1 + \pi_2^*\Delta\phi_2 + \pi_1\Delta\phi_1^* + \pi_2\Delta\phi_2^*)
\end{aligned}
$$

If we set

$$\Phi := \begin{pmatrix} \phi_1 \\ \\ \phi_2 \end{pmatrix}$$

we rewrite our theory as

$$\mathcal{L} = (\partial_\mu \Phi)^\dagger (\partial_\mu \Phi) - m^2 \Phi^\dagger \Phi$$
$$Q = \int d^3x (\dot{\Phi}^\dagger \Delta\Phi + (\Delta\Phi)^\dagger \dot{\Phi})$$

The symmetry of this lagrangian is

$$\Phi \mapsto M\Phi$$

for $M \in U(2)$. We know that this system should have $U(1)$ symmetry from the above problem. Using the $\det : U(n) \to U(1)$ map, we have a short exact sequence

$$SU(2) \to U(2) \to U(1)$$

giving us $U(2) = SU(2) \times U(1)$. For this reason, since $U(1)$ is just a complex number, each conserved charge from this symmetry has the commutation relations of $SU(2)$. In order to put this into a continuous symmetry picture, we exponentiate an element $\sigma \in SU(2)$ is a factor $i(\alpha_1, \alpha_2)$ and take $\alpha_1, \alpha_2 \to 1$:

$$\Phi \mapsto e^{i(\alpha_1,\alpha_2)\sigma}\Phi$$
$$\Delta\Phi \mapsto i\sigma\Phi$$
$$\Delta\Phi^* \mapsto -i\sigma\Phi$$

$SU(2)$ is generated by the Pauli matrices, so we have conserved charges

$$\begin{aligned} Q^i &= i \int d^3x (\dot{\Phi}^\dagger \sigma^i \Phi - \Phi^\dagger \sigma^i \dot{\Phi}) \\ &= i \int d^3x (\phi_a^* \sigma_{ab}^i \pi_b^* - \pi_a \sigma_{ab}^i \phi_b) \end{aligned}$$

Generalizing to n independent identical complex scalar fields, we let $\Phi = (\phi_1, ..., \phi_n)^T$, and our symmetry becomes $U(n) = SU(n) \times U(1)$, meaning the charges we get are of the same form as what we got, but replacing the σ^i with $n-$dimensional skew-hermitian matrices.

□

Question 7. *Evaluate the function*

$$\langle 0| \phi(x)\phi(y) |0\rangle = D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot(x-y)}$$

for $(x-y)$ spacelike so that $(x-y)^2 = -r^2$, explicitly in terms of Bessel functions.

Proof. We have

$$\begin{aligned} D(x-y) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_{\mathbf{p}}} e^{-ip\cdot(x-y)} \\ &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} \frac{dp}{(2\pi)^3} \frac{p^2}{\sqrt{p^2+m^2}} e^{ipr\cos\theta} \end{aligned}$$

θ is the angle between p and $(x-y)$, which also works for the conversion to spherical coordinates. We then have

$$\begin{aligned} D(x-y) &= \frac{1}{(2\pi)^2} \int_0^{\infty} \frac{dpp^2}{\sqrt{p^2+m^2}} \int_0^{\pi} d\theta \sin\theta (\sum_{n=-\infty}^{\infty} J_n(pr) e^{in\theta}) \\ &= \frac{1}{(2\pi)^2} \int_0^{\infty} \frac{dpp^2}{\sqrt{p^2+m^2}} \int_0^{\pi} d\theta \sin\theta (J_0(pr) + 2\sum_{n=1}^{\infty} i^n J_n(pr)\cos(n\theta)) \\ &= \frac{1}{(2\pi)^2} \int_0^{\infty} \frac{dpp^2}{\sqrt{p^2+m^2}} [2J_0(pr) + 2\sum_{n=1}^{\infty} i^n J_n(pr) \frac{\cos n\pi + 1}{1-n^2}] \\ &= \frac{1}{2\pi^2} \int_0^{\infty} \frac{dpp^2}{\sqrt{p^2+m^2}} [J_0(pr) + \sum_{n=1}^{\infty} J_{2n}(pr) \frac{2}{1-4n^2}] \end{aligned}$$

As in the book, the integrand has branch cuts on the imaginary axis starting at $p = \pm im$, so we push the contour up to wrap around the upper branch cut. With $\rho = -ip$, we get

$$D(x-y) = \frac{-i}{2\pi^2}\int_m^\infty d\rho \frac{-\rho^2}{\rho^2 - m^2}[J_0(i\rho r) + \sum_{n=1}^{\infty} J_{2n}(i\rho r)\frac{2}{1-4n^2}]$$

□

Question 8. *Recall the Lorentz commutation relations,*

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho})$$

1. *Define the generators of rotations and boosts as*

$$L^i = \frac{1}{2}\epsilon^{ijk}J^{jk}, \ K^i = J^{0i},$$

where $i, j, k = 1, 2, 3$. An infinitesimal Lorentz transformation can then be written $\Phi \to (1-i\theta$ L-$i\beta \cdot \mathbf{K})\Phi$. Write the commutation relations of these vector operators explicitly. (For example, $[L^i, L^j] = i\epsilon^{ijk}L^k$.) Show that the combinations

$$\mathbf{J}_+ = \frac{1}{2}(\mathbf{L} + i\mathbf{K}) \text{ and } \mathbf{J}_- = \frac{1}{2}(\mathbf{L} - i\mathbf{K})$$

commute with one another and separately satisfy the commutation relations of angular momentum.

2. *The finite-dimensional representations of the rotation group correspond precisely to the allowed values for angular momentum: integers or half-integers. The result of part (a) implies that all finite-dimensional representations of the Lorentz group correspond to pairs of integers or half integers, (j_+, j_-), corresponding to pairs of representations of the rotation group. Using the fact that $\mathbf{J} = \sigma/2$ in the spin-1/2 representation of angular momentum, write explicitly the transformation laws of the 2-component objects transforming according to the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of the Lorentz group. Show that these correspond precisely to the transformations of ψ_L and ψ_R given by*

$$\psi_L \to (1 - i\theta \cdot \frac{\sigma}{2} - \beta \cdot \frac{\sigma}{2})\psi_L;$$
$$\psi_R \to (1 - i\theta \cdot \frac{\sigma}{2} + \beta \cdot \frac{\sigma}{2})\psi_R;$$

3. The identity $\sigma^T = -\sigma^2\sigma\sigma^2$ allows us to rewrite the ψ_L transformation in the unitarily equivalent form

$$\psi' \to \psi'(1 + i\theta \cdot \frac{\sigma}{2} + \beta \cdot \frac{\sigma}{2}),$$

where $\psi' = \psi_L^T\sigma^2$. Using this law, we can represent the object that transforms as $(\frac{1}{2}, \frac{1}{2})$ as a 2×2 matrix that has the ψ_R transformation law on the left and, simultaneously, the transposed ψ_L transformation on the right. Parametrize this matrix as

$$\begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix}$$

Show that the object V^μ transforms as a 4-vector.

Proof. 1. First we calculate $[L^i, L^j]$. We assume that the metric is $(+, -, -, -)$, as per the convention in Peskin and Schroeder.

$$\begin{aligned}
[L^i, L^j] &= \frac{1}{4}(\epsilon^{imn}J^{mn}\epsilon^{jlk}J^{lk} - \epsilon^{jlk}J^{lk}\epsilon^{imn}J^{mn}) \\
&= \frac{1}{4}\epsilon^{imn}\epsilon^{jlk}[J^{mn}, J^{lk}] \\
&= \frac{1}{4}\epsilon^{imn}\epsilon^{jlk}i(g^{nl}J^{mk} - g^{ml}J^{nk} - g^{nk}J^{ml} + g^{mk}J^{nl}) \\
&= \frac{1}{4}\epsilon^{imn}\epsilon^{jlk}i(-\delta^{nl}J^{mk} + \delta^{ml}J^{nk} + \delta^{nk}J^{ml} - \delta^{mk}J^{nl}) \\
&= i\frac{1}{4}(\epsilon^{iml}\epsilon^{jkn}J^{mk} + \epsilon^{imn}\epsilon^{jmk}J^{nk} + \epsilon^{imn}\epsilon^{jln}J^{ml} + \epsilon^{imn}\epsilon^{jml}J^{nl}) \\
\text{(rewrite indices)} &= i(\epsilon^{iml}\epsilon^{jkl}J^{mk}) \\
&= i(\delta_{ij}\delta_{mk} - \delta_{ik}\delta_{jm})J^{mk} \\
&= i(\delta_{jj}J^{mm} - J^{ji}) \\
&= -iJ^{ji}
\end{aligned}$$

Since $J^{ij} = -i(x^i\nabla^j - x^j\nabla^i)$, we have $J^{ij} = -J^{ji}$. Notice that $L^k = \frac{1}{2}\epsilon^{kij}J^{ij}$, so $J^{ij} = -\epsilon^{ijk}L^k$, so $[L^i, L^j] = iJ^{ij} = i\epsilon^{ijk}L^k$.

Similarly, we look at $[K^i, K^j] = [J^{0i}, J^{0j}]$.

$$\begin{aligned}
[K^i, K^j] &= i(g^{i0}J^{0j} - g^{00}J^{ij} - g^{ij}J^{00} + g^{0j}J^{i0}) \\
&= i(-J^{ij} + \delta^{ij}J^{00}) \\
&= -iJ^{ij} \\
&= -i\epsilon^{ijk}L^k
\end{aligned}$$

Now we examine $[L^i, K^j]$:

$$\begin{aligned}
[L^i, K^j] &= [\frac{1}{2}\epsilon^{imk}J^{mk}, J^{0j}] \\
&= \frac{1}{2}\epsilon^{imk}[J^{mk}, J^{0j}] \\
&= \frac{i}{2}\epsilon^{imk}(g^{k0}J^{mj} - g^{m0}J^{kj} - g^{kj}J^{m0} + g^{mj}J^{k0}) \\
&= \frac{i}{2}\epsilon^{imk}(\delta_{k0}J^{mj} - \delta_{m0}J^{kj} + \delta_{kj}J^{m0} - \delta_{mj}J^{k0}) \\
&= \frac{i}{2}\epsilon^{imk}(\delta_{kj}J^{m0} - \delta_{mj}J^{k0}) \\
&= \frac{i}{2}(\epsilon^{imj}J^{m0} - \epsilon^{ijk}J^{k0}) \\
&= \frac{i}{2}(\epsilon^{imj}J^{m0} + \epsilon^{ikj}J^{k0}) \\
&= i\epsilon^{imj}J^{m0} \\
&= -i\epsilon^{imj}J^{0m} \\
&= i\epsilon^{ijm}K^m
\end{aligned}$$

Knowing this, the commutation relations for $\mathbf{J}_\pm$ are:

$$\begin{aligned}
[J_+^i, J_+^j] &= [\frac{1}{2}(L^i + iK^i), \frac{1}{2}(L^j + iK^j)] \\
&= \frac{1}{4}[L^i, L^j] + \frac{i}{4}[L^i, K^j] + \frac{i}{4}[K^i, L^j] - \frac{1}{4}[K^i, K^j] \\
&= \frac{1}{4}i\epsilon^{ijk}L^k + \frac{i}{4}i\epsilon^{ijk}K^k - \frac{i}{4}i\epsilon^{jik}K^k + \frac{1}{4}i\epsilon^{ijk}L^k \\
&= \frac{1}{2}i\epsilon^{ijk}(L^k + iK^k) \\
[J_-^i, J_-^j] &= [\frac{1}{2}(L^i - iK^i), \frac{1}{2}(L^j - iK^j)] \\
&= \frac{1}{4}[L^i, L^j] - \frac{i}{4}[L^i, K^j] - \frac{i}{4}[K^i, L^j] - \frac{1}{4}[K^i, K^j] \\
&= \frac{1}{4}i\epsilon^{ijk}L^k - \frac{i}{4}i\epsilon^{ijk}K^k + \frac{i}{4}i\epsilon^{jik}K^k + \frac{1}{4}i\epsilon^{ijk}L^k \\
&= \frac{1}{2}i\epsilon^{ijk}(L^k - iK^k) \\
[J_+^i, J_-^j] &= [\frac{1}{2}(L^i + iK^i), \frac{1}{2}(L^j - iK^j)] \\
&= \frac{1}{4}[L^i, L^j] - \frac{i}{4}[L^i, K^j] + \frac{i}{4}[K^i, L^j] + \frac{1}{4}[K^i, K^j] \\
&= \frac{1}{4}i\epsilon^{ijk}L^k - \frac{i}{4}i\epsilon^{ijk}K^k - \frac{i}{4}i\epsilon^{jik}K^k - \frac{1}{4}i\epsilon^{ijk}L^k \\
&= 0
\end{aligned}$$

2. In the $(j_+, j_-) = (\frac{1}{2}, 0)$ case, we have $\mathbf{J}_+ = \frac{\sigma}{2}$ in the spin$-\frac{1}{2}$ representation of the rotation group. To ensure this, we set $\mathbf{L} = \sigma, \mathbf{K} = -i\sigma$. Similarly, for the (j_+, j_-) case, we set $L = \sigma, \mathbf{K} = i\sigma$. The Lorentz operators are given by $exp[-i\frac{1}{2}(\theta\mathbf{L} \pm i\beta\mathbf{K})]$ for $\mathbf{J}_\pm$. Taylor expanding for infinitesimal θ, β, and neglecting >1 order terms of θ and β, the infinitesimal transformations are given by

$$\Phi \to (1 - i\theta \cdot \frac{\sigma}{2} \mp \beta \cdot \frac{\sigma}{2})\Phi$$

3. Notice that

$$\begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix} = V^\mu \sigma_\mu$$

Thus the field is given by $\psi_R V^\mu \sigma_\mu \psi_L^T \sigma^2$, and the field transforms by

$$\psi_R V^\mu \sigma_\mu \psi_L^T \sigma^2 \to (1 - i\theta \cdot \frac{\sigma}{2} + \beta \cdot \frac{\sigma}{2})\psi_R V^\mu \sigma_\mu \psi_L^T \sigma^2 (1 + i\theta \cdot \frac{\sigma}{2} + \beta \cdot \frac{\sigma}{2})$$

Expanding this out, we have this equal to

$$\begin{aligned}(1 - \frac{i}{2}\theta^j\theta^j + \frac{1}{2}\beta^j\theta^j)(V^0 + V^i\sigma^i)(1 + \frac{i}{2}\theta^j\theta^j + \frac{1}{2}\beta^j\theta^j) &= (1 - \frac{i}{2}\theta^j\theta^j + \frac{1}{2}\beta^j\theta^j)V^0 \\ &+ V^0(1 + \frac{i}{2}\theta^j\theta^j + \frac{1}{2}\beta^j\theta^j) \\ &+ (V^i\sigma^i - \frac{i}{2}\theta^i V^i\sigma^j\sigma^i + \frac{1}{2}\beta^j V^i\sigma^j\sigma^i) \\ &+ (V^i\sigma^i + \frac{i}{2}V^i\theta^j\sigma^i\sigma^j + \frac{1}{2}V^i\beta^j\sigma^i\sigma^j) \\ &= (1 + \beta^j\sigma^j)V^0 \\ &+ 2V^i\sigma^i + \frac{1}{2}\beta^j V^i\{\sigma^j, \sigma^i\} + \frac{i}{2}\theta^j V^i[\sigma^i, \sigma^j] \\ &= (1 + \beta^j\sigma^j)V^0 \\ &+ 2V^i\sigma^i + \frac{1}{2}\beta^j V^i 2\delta^i_j + \frac{i}{2}\theta^j V^i 2i\epsilon^{ijk}\sigma^k \\ &= (1 + \beta^j\sigma^j)V^0 \\ &+ (2\sigma^j - \theta^j\epsilon^{ijk}\sigma^k + \beta^i)V^i\end{aligned}$$

This is a Lorentz boost, so $V^0 + V^i = V^\mu$ is a Lorentz 4-vector.

□

Question 9. *Derive the Gordon identity,*

$$\overline{u}(p')\gamma^\mu u(p) = \overline{u}(p')[\frac{(p')^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}]u(p),$$

where $q = (p' - p)$.

Proof. We see a $i\sigma^{\mu\nu}$ in there, so we'll probably want to establish some $[\gamma^\mu, \gamma^\nu]$ in there, since this commutator is equal to $-2i\sigma^{\mu\nu}$. In equation 3.46 in Peskin & Schroeder, we have, from the Dirac Equation,

$$(\gamma^\mu p_\mu - m)u(p) = 0$$

If we try to come up with a similar equation with the adjoint $u^\dagger(p')$, if we throw in a γ^0 we get the relation:

$$u^\dagger(p')\gamma^0(\gamma_\mu(p')^\mu - m) = 0$$
$$\overline{u}(p')(\gamma_\mu(p')^\mu - m) = 0$$

Thus we slip in an m factor to consider $\overline{u}(p')\gamma^\mu m u(p)$ have

$$\overline{u}(p')\gamma^\mu m u(p) = \overline{u}(p')\gamma^\mu\gamma^\nu p_\nu u(p)$$
$$\overline{u}(p')m\gamma^\mu u(p) = \overline{u}(p')\gamma^\nu p'_\nu \gamma^\mu u(p)$$

Adding these two equations, we get

$$\overline{u}(p')2m\gamma^\mu u(p) = \overline{u}(p')(\gamma^\mu\gamma^\nu p_\nu + \gamma^\nu\gamma^\mu p'_\mu)u(p)$$

We can express this in two ways, one where there is only $\gamma^\mu\gamma^\nu$ and one where there is only $\gamma^\nu\gamma^\mu$:

$$\overline{u}(p')2m\gamma^\mu u(p) = \overline{u}(p')(\gamma^\mu\gamma^\nu p_\nu + \gamma^\mu\gamma^\nu p'_\mu + 2i\sigma^{\mu\nu}p'_\mu)u(p),$$
$$\overline{u}(p')2m\gamma^\mu u(p) = \overline{u}(p')(\gamma^\nu\gamma^\mu p_\nu + \gamma^\nu\gamma^\mu p'_\mu - 2i\sigma^{\mu\nu}p_\nu)u(p)$$

Adding these together, we get

$$\overline{u}(p')4m\gamma^\mu u(p) = \overline{u}(p')(\{\gamma^\mu, \gamma^\nu\}(p_\nu + p'_\mu) + 2i\sigma^{\mu\nu}(p'_\mu - p_\mu))u(p)$$

□

Question 10. *Let k_0^μ, k_1^ν be fixed 4-vectors satisfying $k_0^2 = 0, k_1^2 = -1, k_0 \cdot k_1 = 0$. Define basic spinors in the following way: Let u_{L0} be the left-handed spinor for a fermion with momentum k_0. Let $u_{R0} = \not{k}_1 u_{L0}$. Then, for any p such that p is lightlike ($p^2 = 0$), define*

$$u_L(p) = \frac{1}{\sqrt{2p \cdot k_0}}\not{p}u_{R0} \text{ and } u_R(p) = \frac{1}{\sqrt{2p \cdot k_0}}\not{p}u_{L0}.$$

1. *Show that* $\not{k}_0 u_{R0} = 0$. *Show that, for any lightlike* p, $\not{p} u_L(p) = \not{p} u_R(P) = 0$.

2. *For the choices* $k_0 = (E, 0, 0, -E), k_1 = (0, 1, 0, 0)$, *construct* $u_{L0}, u_{R0}, u_L(p)$, *and* $u_R(p)$ *explicitly.*

3. *Define the spinor products* $s(p_1, p_2)$ *and* $t(p_1, p_2)$, *for* p_1, p_2 *lightlike, by*

$$s(p_1, p_2) = \overline{u}_R(p_1) u_L(p_2), \;\; t(p_1, p_2) = \overline{u}_L(p_1) u_R(p_2).$$

   Using the explicit forms for the u_λ *given in part 2, compute the spinor products explicitly and show that* $t(p_1, p_2) = (s(p_2, p_1))^*$ *and* $s(p_1, p_2) = -s(p_2, p_1)$. *In addition, show that*

$$|s(p_1, p_2)|^2 = 2 p_1 \cdot p_2.$$

   Thus the spinor products are the square roots of 4-vectors dot products.

Proof. With the definition, we have

$$\begin{aligned} \not{k}_0 \not{k}_1 u_{L0} &= \gamma^\mu (k_0)_\mu \gamma^\nu (k_1)_\nu u_{L0} \\ &= [2 g^{\mu\nu} - \gamma^\nu \gamma^\mu](k_0)_\mu (k_1)_\nu u_{L0} \\ &= k_0 \cdot k_1 u_{L0} - \gamma^\nu \gamma^\mu (k_0)_\mu (k_1)_\nu u_{L0} \\ &= 0 - \not{k}_0 \not{k}_1 u_{L0} \end{aligned}$$

1. Thus $\not{k}_0 \not{k}_1 u_{L0}$ must be 0.

   For any lightlike p, the computation follows the same way.

$$\begin{aligned} \not{p} u_L(p) &= \frac{1}{\sqrt{2p \cdot k_0}} \not{p} \not{p} \not{k}_1 u_{L0} \\ &= \frac{1}{\sqrt{2p \cdot k_0}} [2 g^{\mu\nu} - \gamma^\nu \gamma^\mu] p_\mu p_\nu \not{k}_1 u_{L0} \\ &= -\frac{1}{\sqrt{2p \cdot k_0}} \not{p} \not{p} \not{k}_1 u_{L0} \end{aligned}$$

   since $g^{\mu\nu} p_\mu p_\nu = 0$ for p lightlike. That $\not{p} u_R(p) = 0$ follows in the same way.

2. u_{L0} is the left Weyl spinor of a fermion with momentum k_0. From 3.50 in Peskin and

Schroeder, we have

$$u^s(k_0) = \begin{pmatrix} \sqrt{k_0 \cdot \sigma}\xi^s \\ \sqrt{k_0 \cdot \overline{\sigma}}\xi^s \end{pmatrix}$$

where $s = 1$ we'll decide corresponds to the left-handed spinor, and $s = 2$ corresponds to the right-handed one. For $k_0 = (E, 0, 0, -E)$, we have

$$u_{L0} = u^1(k_0) = \begin{pmatrix} \sqrt{\begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix}}\xi^1 \\ \sqrt{\begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix}}\xi^1 \end{pmatrix} = \sqrt{2E}\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\xi^1 \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\xi^1 \end{pmatrix}$$

Multiplying this by $\not{k}_1$ to get u_{R0}, we have

$$\not{k}_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\to u_{R0} = \not{k}_1 u_{L0} = \sqrt{2E}\begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\xi^1 \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\xi^1 \end{pmatrix} = \sqrt{2E}\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\xi^2 \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\xi_2 \end{pmatrix}$$

For $\xi^1 := (\xi_1, \xi_2)^T$, we therefore have $\xi^2 = (-\xi_2, \xi_1)^T$, and thus

$$u_{L0} = \sqrt{2E}\begin{pmatrix} \xi_1 \\ 0 \\ 0 \\ \xi_2 \end{pmatrix}, u_{R0} = \sqrt{2E}\begin{pmatrix} -\xi_2 \\ 0 \\ 0 \\ \xi_1 \end{pmatrix}$$

Multiplying these by $\frac{1}{\sqrt{2p\cdot k_0}}\not{p} = \frac{1}{\sqrt{2p\cdot k_0}}\begin{pmatrix} 0 & 0 & p_0 - p_3 & -p_1 + ip_2 \\ 0 & 0 & -p_1 - ip_2 & p_0 + p_3 \\ p_0 + p_3 & p_1 - ip_2 & 0 & 0 \\ p_1 + ip_2 & p_0 - p_3 & 0 & 0 \end{pmatrix}$ to get $u_L(p)$ and $u_R(p)$, respectively, we get:

$$u_L(p) = \frac{1}{\sqrt{p_0 + p_3}}\begin{pmatrix} (-p_1 + ip_2)\xi_1 \\ (p_0 + p_3)\xi_1 \\ -(p_0 + p_3)\xi_2 \\ -(p_1 + ip_2)\xi_2 \end{pmatrix}, u_R(p) = \frac{1}{\sqrt{p_0 + p_3}}\begin{pmatrix} (-p_1 + ip_2)\xi_2 \\ (p_0 + p_3)\xi_2 \\ (p_0 + p_3)\xi_1 \\ (p_1 + ip_2)\xi_1 \end{pmatrix}$$

3. Writing out $s(p_1, p_2)$, we have

$$\frac{1}{\sqrt{p_0^{(1)} + p_3^{(1)}}}\begin{pmatrix} (-p_1^{(1)} - ip_2^{(1)})\xi_2^* \\ (p_0^{(1)} + p_3^{(1)})\xi_2^* \\ (p_0^{(1)} + p_3^{(1)})\xi_1^* \\ (p_1^{(1)} - ip_2^{(1)})\xi_1^* \end{pmatrix}^T \begin{pmatrix} 0 & Id \\ Id & 0 \end{pmatrix} \frac{1}{\sqrt{p_0^{(2)} + p_3^{(2)}}}\begin{pmatrix} (-p_1^{(2)} + ip_2^{(2)})\xi_1 \\ (p_0^{(2)} + p_3^{(2)})\xi_1 \\ -(p_0^{(2)} + p_3^{(2)})\xi_2 \\ -(p_1^{(2)} + ip_2^{(2)})\xi_2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{p_0^{(1)} + p_3^{(1)}}}\frac{1}{\sqrt{p_0^{(2)} + p_3^{(2)}}}[(p_0^{(1)} + p_3^{(1)})(-p_1^{(2)} + ip_2^{(2)})\xi_1^*\xi_1 + (p_0^{(2)} + p_3^{(2)})(p_1^{(1)} - ip_2^{(1)})\xi_1^*\xi_1$$

$$+(p_0^{(2)} + p_3^{(2)})(p_1^{(1)} + ip_2^{(1)})\xi_2^*\xi_2 - (p_0^{(1)} + p_3^{(1)})(p_1^{(2)} + ip_2^{(2)})\xi_2^*\xi_2]$$

For the last expression, the two terms in the bracket on the top line are of opposite sign if $p^{(1)}$ and $p^{(0)}$ are swapped, and the same goes for the bottom line. Thus $s(p_1, p_2) = -s(p_2, p_1)$.

$t(p_1, p_2)$ is calculated in the same way:

$$\frac{1}{\sqrt{p_0^{(1)}+p_3^{(1)}}}\begin{pmatrix}(-p_1^{(1)}-ip_2^{(1)})\xi_1^*\\(p_0^{(1)}+p_3^{(1)})\xi_1^*\\-(p_0^{(1)}+p_3^{(1)})\xi_2^*\\(-p_1^{(1)}+ip_2^{(1)})\xi_2^*\end{pmatrix}^T\begin{pmatrix}0 & Id\\ Id & 0\end{pmatrix}\frac{1}{\sqrt{p_0^{(2)}+p_3^{(2)}}}\begin{pmatrix}(-p_1^{(2)}+ip_2^{(2)})\xi_2\\(p_0^{(2)}+p_3^{(2)})\xi_2\\(p_0^{(2)}+p_3^{(2)})\xi_1\\(p_1^{(2)}+ip_2^{(2)})\xi_1\end{pmatrix}$$

$$= \frac{1}{\sqrt{p_0^{(1)}+p_3^{(1)}}}\frac{1}{\sqrt{p_0^{(2)}+p_3^{(2)}}}[-(p_0^{(1)}+p_3^{(1)})(-p_1^{(2)}+ip_2^{(2)})\xi_2^*\xi_2+(p_0^{(2)}+p_3^{(2)})(-p_1^{(1)}+ip_2^{(1)})\xi_2^*\xi_2$$
$$+(p_0^{(2)}+p_3^{(2)})(-p_1^{(1)}-ip_2^{(1)})\xi_1^*\xi_1+(p_0^{(1)}+p_3^{(1)})(p_1^{(2)}+ip_2^{(2)})\xi_1^*\xi_1]$$

Swapping $p^{(1)}$ and $p^{(2)}$ and taking the complex conjugate of this, we get

$$\frac{1}{\sqrt{p_0^{(1)}+p_3^{(1)}}}\frac{1}{\sqrt{p_0^{(2)}+p_3^{(2)}}}[-(p_0^{(2)}+p_3^{(2)})(-p_1^{(1)}-ip_2^{(1)})\xi_2^*\xi_2+(p_0^{(1)}+p_3^{(1)})(-p_1^{(2)}-ip_2^{(2)})\xi_2^*\xi_2$$
$$+(p_0^{(1)}+p_3^{(1)})(-p_1^{(2)}+ip_2^{(2)})\xi_1^*\xi_1+(p_0^{(2)}+p_3^{(2)})(p_1^{(1)}-ip_2^{(1)})\xi_1^*\xi_1]$$

which is equal to $s(p_1, p_2)$ above.

We write $|s(p_1,p_2)|^2 = s(p_1,p_2)s(p_1,p_2)^*$ as

$$\frac{1}{\sqrt{p_0^{(1)}+p_3^{(1)}}}\frac{1}{\sqrt{p_0^{(2)}+p_3^{(2)}}}[(A+B)\xi_1^*\xi_1+(A^*+B^*)\xi_2^*\xi_2]$$
$$\times\frac{1}{\sqrt{p_0^{(1)}+p_3^{(1)}}}\frac{1}{\sqrt{p_0^{(2)}+p_3^{(2)}}}[(A^*+B^*)\xi_1^*\xi_1+(A+B)\xi_2^*\xi_2]$$

where

$$A = (p_0^{(1)}+p_3^{(1)})(-p_1^{(2)}+ip_2^{(2)})$$
$$B = (p_0^{(2)}+p_3^{(2)})(p_1^{(1)}-ip_2^{(1)})$$

Giving us

$$\frac{[|A+B|^2(\xi_1^*\xi_1\xi_1^*\xi_1+\xi_2^*\xi_2\xi_2^*\xi_2)+[(A+B)^2+(A^*+B^*)^2]\xi_1^*\xi_1\xi_2^*\xi_2]}{(p_0^{(1)}+p_3^{(1)})(p_0^{(2)}+p_3^{(2)})}$$
$$=\frac{[|A+B|^2((\xi_1^*\xi_1+\xi_2^*\xi_2)^2-2\xi_1^*\xi_1\xi_2^*\xi_2)+[(A+B)^2+(A^*+B^*)^2]\xi_1^*\xi_1\xi_2^*\xi_2]}{(p_0^{(1)}+p_3^{(1)})(p_0^{(2)}+p_3^{(2)})}$$

We assume ξ is normalized, so we first examine $\frac{|A+B|^2}{(p_0^{(1)}+p_3^{(1)})\cdot(p_0^{(2)}+p_3^{(2)})}$:

$$\begin{aligned}|A+B|^2 &= ((p_0^{(2)}+p_3^{(2)})p_1^{(1)}-(p_0^{(1)}+p_3^{(1)})p_1^{(2)})^2\\ &+((p_0^{(1)}+p_3^{(1)})p_1^{(2)}-(p_0^{(2)}+p_3^{(2)})p_1^{(1)})^2\\ &=(p_0^{(2)}+p_3^{(2)})^2(p_1^{(1)}p_1^{(1)}+p_2^{(1)}p_2^{(1)})+(p_0^{(1)}+p_3^{(1)})^2(p_1^{(2)}p_1^{(2)}+p_2^{(2)}p_2^{(2)})\\ &-2(p_0^{(2)}+p_3^{(2)})(p_0^{(1)}+p_3^{(1)})(p_1^{(1)}p_1^{(2)}+p_2^{(2)}p_2^{(1)})\end{aligned}$$

Dividing by $(p_0^{(1)}+p_3^{(1)})(p_0^{(2)}+p_3^{(2)})$, we get

$$\begin{aligned}\frac{|A+B|^2}{(p_0^{(1)}+p_3^{(1)})(p_0^{(2)}+p_3^{(2)})} &= \frac{(p_0^{(2)}+p_3^{(2)})}{(p_0^{(1)}+p_3^{(1)})}(p_1^{(1)}p_1^{(1)}+p_2^{(1)}p_2^{(1)})\\ &+\frac{(p_0^{(1)}+p_3^{(1)})}{(p_0^{(2)}+p_3^{(2)})}(p_1^{(2)}p_1^{(2)}+p_2^{(2)}p_2^{(2)})\\ &-2(p_1^{(2)}p_1^{(2)}+p_2^{(2)}p_2^{(1)})\end{aligned}$$

Noting that $(p_1^{(s)}p_1^{(s)}+p_2^{(s)}p_2^{(s)})=(p_0^{(s)}p_0^{(s)}-p_3^{(s)}p_3^{(s)})=(p_0^{(s)}-p_3^{(s)})(p_0^{(s)}+p_3^{(s)})$, since p_s is lightlike for $s \in \{1,2\}$, gives

$$
\begin{aligned}
\frac{|A+B|^2}{(p_0^{(1)}+p_3^{(1)})(p_0^{(2)}+p_3^{(2)})} &= \frac{(p_0^{(2)}+p_3^{(2)})}{(p_0^{(1)}+p_3^{(1)})}(p_0^{(1)}p_0^{(1)} - p_3^{(1)}p_3^{(1)}) \\
&+ \frac{(p_0^{(1)}+p_3^{(1)})}{(p_0^{(2)}+p_3^{(2)})}(p_0^{(2)}p_0^{(2)} - p_3^{(2)}p_3^{(2)}) \\
&- 2(p_1^{(2)}p_1^{(2)} + p_2^{(2)}p_2^{(1)}) \\
&= (p_0^{(2)}+p_3^{(2)})(p_0^{(1)}-p_3^{(1)}) \\
&+ (p_0^{(1)}+p_3^{(1)})(p_0^{(2)}-p_3^{(2)}) \\
&- 2(p_1^{(2)}p_1^{(2)} + p_2^{(2)}p_2^{(1)}) \\
&= 2p_0^{(1)}p_0^{(2)} - 2p_1^{(1)}p_1^{(2)} - 2p_2^{(1)}p_2^{(2)} - 2p_3^{(1)}p_3^{(2)} \\
&= 2p^{(1)} \cdot p^{(2)}
\end{aligned}
$$

Things get hairy if we examine $\frac{(A+B)^2+(A^*+B^*)^2-2|A+B|^2}{(p_0^{(1)}+p_3^{(1)})(p_0^{(2)}+p_3^{(2)})}\xi_1^*\xi_1\xi_2^*\xi_2$. We get the proof that $|s(p_1,p_2)|^2 = 2p_1 \cdot p_2$ if either ξ_1 or ξ_2 is equal to 0, but if both are nonzero, this is not the case:

$$
\begin{aligned}
A^2 &+ 2AB + B^2 + (A^*)^2 + 2A^*B^* + (B^*)^2 - 2(A+B)(A^*+B^*) \\
&= A^2 + 2AB + B^2 + (A^*)^2 + 2A^*B^* + (B^*)^2 - 2AA^* - 2AB^* - 2A^*B - 2BB^* \\
&= (A-A^*)^2 + (B-B^*)^2 + 2[AB + A^*B^* - AB^* - A^*B] \\
&= (2Im(A))^2 + (2Im(B))^2 + 2[A(B-B^*) + A^*(B^*-B)] \\
&= 4[Im(A)^2 + Im(B)^2] + 2[A2Im(B) - A^*2Im(B)] \\
&= 4[Im(A)^2 + Im(B)^2 + 2Im(B)Im(A)] \\
&= 4(Im(A)+Im(B))^2 \\
&= 4[(p_0^{(1)}+p_3^{(1)})ip_2^{(2)} - (p_0^{(2)}+p_3^{(2)})ip_2^{(1)})]^2
\end{aligned}
$$

If, say, $p_1 = (X,X,0,0)$ and $p_2 = (Y,0,Y,0)$, this is nonzero. For now, though, we can set $\xi = (1,0)^T$, and complete the proof.

□

Question 11. *Recall that one can write a relativistic equation for a massless 2-component fermion field that transforms as the upper two components of a Dirac spinor (ψ_L). Call such a 2-component field $\chi_a(x), a = 1, 2$.*

1. *Show that it is possible to write an equation for $\chi(x)$ as a massive field in the following way:*

$$i\overline{\sigma} \cdot \partial\chi - im\sigma^2\chi^* = 0$$

That is, show, first, that this equation is relativistically invariant and, second, that it implies the Klein-Gordon equation, $(\partial^2 + m^2)\chi = 0$. This form of the fermion mass is called a Majorana mass term.

2. *Grassmann numbers α, β satisfy $\alpha\beta = -\beta\alpha$. A Grassmann field $\xi(x)$ can be expanded in a basis of functions as*

$$\xi(x) = \sum_n \alpha_n \phi_n(x)$$

where the $\phi_n(x)$ are orthogonal c-number functions and the α_n are a set of independent Grassmann numbers. Define the complex conjugate of a product of Grassmann numbers to reverse the order:

$$(\alpha\beta)^* := \beta^*\alpha^* = -\alpha^*\beta^*.$$

Show that the classical action,

$$S = \int d^4x[\chi^\dagger i\overline{\sigma} \cdot \partial\chi + \frac{im}{2}(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*)],$$

(where $\chi^\dagger = (\chi^*)^T$) is real $S^* = S$), and that varying this S with respect to χ and χ^* yields the Majorana equation.

3. *We can rewrite the 4-component Dirac field in terms of two 2-component spinors:*

$$\psi_L(x) = \chi_1(x), \psi_R(x) = i\sigma^2\chi_2^*(x)$$

Rewrite the Dirac Lagrangian in terms of χ_1 and χ_2 and note the form of the mass term.

4. *Show that the previous action has a global symmetry. Compute the divergences of the currents*

$$J^\mu = \chi^\dagger \overline{\sigma}^\mu \chi, \; J^\mu = \chi_1^\dagger \overline{\sigma}^\mu \chi_1 - \chi_2^\dagger \overline{\sigma}^\mu \chi_2,$$

for the theories of parts 2 and 3 of this question, respectively, and relate your results to the symmetries of these theories. Comstruct a theory of N free massive 2-component fermion fields with $O(n)$ symmetry.

5. *Quantize the theory of parts 1) and 2). HINT: Compare the top two indices of the quantized Dirac field.*

Proof. 1. Transforming this expression by a Lorentz boost gives

$$\begin{aligned} i\overline{\sigma} \cdot \partial \chi(x) - im\sigma^2 \chi^*(x) &\longrightarrow i\overline{\sigma} \cdot \partial \chi(\Lambda^{-1}x) - im\sigma^2 \chi^*(\Lambda^{-1}x) \\ &= i\overline{\sigma}^\nu g_{\mu\nu} (\Lambda^{-1})^\mu_\lambda \partial^\lambda \Lambda_{\frac{1}{2}} \chi(\Lambda^{-1}x) - im\sigma^2 \Lambda_{\frac{1}{2}} \chi^*(\Lambda^{-1}x) \\ &= \Lambda_{\frac{1}{2}} \Lambda_{\frac{1}{2}}^{-1} i\overline{\sigma}^\nu \Lambda_{\frac{1}{2}} g_{\mu\nu} (\Lambda^{-1})^\mu_\lambda \partial^\lambda \chi(\Lambda^{-1}x) \\ &- im\Lambda_{\frac{1}{2}} \Lambda_{\frac{1}{2}}^{-1} \sigma^2 \Lambda_{\frac{1}{2}} \chi^*(\Lambda^{-1}x) \end{aligned}$$

Noting that σ is a 2×2-representation of the Dirac algebra, we have $\Lambda_{\frac{1}{2}}^{-1} \overline{\sigma}^\nu \Lambda_{\frac{1}{2}} = \Lambda^\nu_\lambda \overline{\sigma}^\lambda$. Thus this last expression is equal to

$$\begin{aligned} \Lambda_{\frac{1}{2}} \Lambda^\nu_\lambda i\overline{\sigma}^\lambda g_{\mu\nu} (\Lambda^{-1})^\mu_\iota \partial^\iota \chi(\Lambda^{-1}x) - im\Lambda_{\frac{1}{2}} \sigma^2 \chi^*(\Lambda^{-1}x) \\ = \Lambda_{\frac{1}{2}} [i\overline{\sigma}^\lambda g_{\iota\lambda} \partial^\iota \chi(\Lambda^{-1}x) - im\sigma^2 \chi^*(\Lambda^{-1}x)] \\ = \Lambda_{\frac{1}{2}} [i\overline{\sigma} \cdot \partial \chi(\Lambda^{-1}x) - im\sigma^2 \chi^*(\Lambda^{-1}x)] \\ = 0 \end{aligned}$$

Thus this equation is relativistically invariant.

We have the system of equations

$$\begin{pmatrix} \partial_0 - \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 + \partial_3 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = m \begin{pmatrix} -i\chi_2^* \\ i\chi_1^* \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} (\partial_0 - \partial_3) & \partial_1 & 0 & \partial_2 \\ \partial_1 & (\partial_0 + \partial_3) & -\partial_2 & 0 \\ 0 & -\partial_2 & (\partial_0 - \partial_3) & \partial_1 \\ \partial_2 & 0 & \partial_1 & (\partial_0 + \partial_3) \end{pmatrix} \begin{pmatrix} Re(\chi_1) \\ Re(\chi_2) \\ iIm(\chi_1) \\ iIm(\chi_2) \end{pmatrix} = m \begin{pmatrix} -Im(\chi_2) \\ Im(\chi_1) \\ -iRe(\chi_2) \\ iRe(\chi_1) \end{pmatrix}$$

Thus we recast $i\overline{\sigma} \cdot \partial\chi - im\sigma^2\chi^* = 0$ as

$$\begin{pmatrix} (\partial_0 - \partial_3) & \partial_1 & 0 & \partial_2 - m \\ \partial_1 & (\partial_0 + \partial_3) & -\partial_2 + m & 0 \\ 0 & -\partial_2 + m & (\partial_0 - \partial_3) & \partial_1 \\ \partial_2 - m & 0 & \partial_1 & (\partial_0 + \partial_3) \end{pmatrix} \begin{pmatrix} Re(\chi_1) \\ Re(\chi_2) \\ iIm(\chi_1) \\ iIm(\chi_2) \end{pmatrix} = 0$$

Call this 4×4 operator M. It is easy to see that

$$\text{diagonal}[\partial^2] / \begin{pmatrix} (\partial_0 - \partial_3) & \partial_1 & 0 & \partial_2 \\ \partial_1 & (\partial_0 + \partial_3) & -\partial_2 & 0 \\ 0 & -\partial_2 & (\partial_0 - \partial_3) & \partial_1 \\ \partial_2 & 0 & \partial_1 & (\partial_0 + \partial_3) \end{pmatrix}$$

$$= \begin{pmatrix} (\partial_0 + \partial_3) & -\partial_1 & 0 & -\partial_2 \\ -\partial_1 & (\partial_0 - \partial_3) & \partial_2 & 0 \\ 0 & \partial_2 & (\partial_0 + \partial_3) & -\partial_1 \\ -\partial_2 & 0 & -\partial_1 & (\partial_0 - \partial_3) \end{pmatrix}$$

Let M^t be defined as

$$\begin{pmatrix} (\partial_0 + \partial_3) & -\partial_1 & 0 & -\partial_2 - m \\ -\partial_1 & (\partial_0 - \partial_3) & \partial_2 + m & 0 \\ 0 & \partial_2 + m & (\partial_0 + \partial_3) & -\partial_1 \\ -\partial_2 - m & 0 & -\partial_1 & (\partial_0 - \partial_3) \end{pmatrix}$$

We then have, acting on both sides of $M\chi = 0$ by M^t,

$$M^t M\chi = \text{diag}[\partial^2 + m^2]\chi = (\partial^2 + m^2)\chi = 0$$

2. In considering the first term, we have, since $i\overline{\sigma} \cdot \partial\chi - im\sigma^2\chi^* = 0$,

$$\begin{aligned} \chi^\dagger i\overline{\sigma} \cdot \partial\chi = \chi^\dagger im\sigma^2\chi^* &= \chi_1^*\chi_2 - \chi_1\chi_2^* \\ &= \chi_1^*\chi_2 + (\chi_1^*\chi_2)^* \\ &= Re(\chi_1^*\chi_2) \end{aligned}$$

We proceed with the second term in the same way:

$$\begin{aligned} \frac{im}{2}(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*) &= \frac{m}{2}(\chi_1\chi_2 - \chi_2\chi_1 - \chi_1^*\chi_2^* + \chi_2^*\chi_1^*) \\ &= \frac{m}{2}(Re(\chi_1\chi_2) - Re(\chi_2\chi_1)) \end{aligned}$$

We expand the Lagrangian to get

$$\begin{aligned} \mathcal{L} = i\chi_1^*(\partial_0 - \partial_3)\chi_1 &+ i\chi_1^*(-\partial_1 + i\partial_2)\chi_2 \\ + i\chi_1^*(-\partial_1 - i\partial_2)\chi_1 &+ i\chi_2^*(\partial_0 + \partial_3)\chi_2 \\ + \frac{im}{2}[-i\chi_1\chi_2 &+ i\chi_2\chi_1 + i\chi_1^*\chi_2^* - i\chi_2^*\chi_1^*] \end{aligned}$$

Varying both χ and χ^*, we get the Euler-Lagrange equations as

$$\frac{\partial \mathcal{L}}{\partial \chi} = \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi)} i \begin{pmatrix} -m\chi_2 \\ m\chi_1 \end{pmatrix} = i \begin{pmatrix} \partial_0 \chi_1^* - \partial_1 \chi_2^* - i\partial_2 \chi_2^* - \partial_3 \chi_1^* \\ \partial_0 \chi_2^* - \partial_1 \chi_1^* + i\partial_2 \chi_1^* + \partial_0 \chi_2^* \end{pmatrix}$$

$$im\sigma^2 \chi^* = i\overline{\sigma} \cdot \partial \chi$$

$$\frac{\partial \mathcal{L}}{\partial \chi^*} = 0$$

$$i \begin{pmatrix} (\partial_0 - \partial_3)\chi_1 + (-\partial_1 + i\partial_2)\chi_2 - m\chi_2^* \\ (-\partial_1 - i\partial_2)\chi_1 + (\partial_0 + \partial_3)\chi_2 - m\chi_1^* \end{pmatrix} = 0$$

$$i\overline{\sigma} \cdot \partial \chi - im\sigma^2 \chi = 0$$

3. In terms of χ_1 and χ_2, we have

$$\begin{aligned}
&\begin{pmatrix} \chi_1^\dagger & -i\chi_2^T \sigma^2 \end{pmatrix} (i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \partial_\mu \\ \overline{\sigma}^\mu \partial_\mu & 0 \end{pmatrix} - m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \begin{pmatrix} \chi_1 \\ i\sigma^2 \chi_2^* \end{pmatrix} \\
&= \begin{pmatrix} \chi_1^\dagger & -i\chi_2^T \sigma^2 \end{pmatrix} (i \begin{pmatrix} \overline{\sigma}^\mu \partial_\mu & 0 \\ 0 & \sigma^\mu \partial_\mu \end{pmatrix} - m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \begin{pmatrix} \chi_1 \\ i\sigma^2 \chi_2^* \end{pmatrix} \\
&= i\chi_1^\dagger \overline{\sigma}^\mu \partial_\mu \chi_1 + i\chi_2^T \sigma^2 \sigma^\mu \sigma^2 \partial_\mu \chi_2^* - \chi_1^\dagger m i \sigma^2 \chi_2^* + \chi_2^T m i \sigma^2 \chi_1 \\
&= i\chi_1^\dagger \overline{\sigma}^\mu \partial_\mu \chi_1 + i\chi_2^T \overline{\sigma}^\mu \partial_\mu \chi_2^* - \chi_1^\dagger m i \sigma^2 \chi_2^* + \chi_2^T m i \sigma^2 \chi_1
\end{aligned}$$

The mass term has the $\chi_{1,2}$ swapped and they are conjugates of each other.

4. The previous part has a global symmetry of $\chi_1 \to e^{i\alpha_1}\chi_1, \chi_2 \to e^{i\alpha_2}\chi_2$. When computing $\partial_\mu(\chi^\dagger \overline{\sigma}^\mu \chi)$, we start with the product rule:

$$\begin{aligned}
\partial_\mu(\chi^\dagger \overline{\sigma}^\mu \chi) &= \partial_\mu \chi^\dagger \overline{\sigma}^\mu \chi + \chi^\dagger \overline{\sigma}^\mu \partial_\mu \chi \\
&= (\chi^\dagger \overline{\sigma}^\mu \partial_\mu \chi)^\dagger + \chi^\dagger \overline{\sigma}^\mu \partial_\mu \chi \\
&= 2Re(\chi^\dagger \overline{\sigma}^\mu \partial_\mu \chi) \\
&= 0
\end{aligned}$$

This is zero because, as shown above, $i \times \chi^\dagger \overline{\sigma}^\mu \partial_\mu \chi$ is real. In the same way, we have

$$\begin{aligned}
\partial_\mu(\chi_1^\dagger \overline{\sigma}^\mu \chi_1 - \chi_2^\dagger \overline{\sigma}^\mu \chi_2) &= 2Re(\chi_1^\dagger \overline{\sigma}^\mu \partial_\mu \chi_1) - 2Re(\chi_2^\dagger \overline{\sigma}^\mu \partial_\mu \chi_2) \\
&= 0 - 2Re((\chi_2^T \overline{\sigma}^\mu \partial_\mu \chi_2^*)^*) \\
&= 0
\end{aligned}$$

These are the Noether conserved Noether currents corresponding to the $\chi \to e^{i\alpha}\chi$ and $\chi_1 \to e^{i\alpha_1}\chi_1, \chi_2 \to e^{i\alpha_2}\chi_2$ symmetries, respectively.

For an N-massive 2-component fermion system with $O(N)$ symmetry, we first note that symmetry is given by multiplying $(\chi_1, ..., \chi_N)^T$, where χ is a 2-component fermion field, by $O(N) \otimes Id_2$, where $\otimes$ is the Kronecker product. The action is then just taken from part 2:

$$S = \int d^4x[\begin{pmatrix} \chi_1 \\ \vdots \\ \chi_N \end{pmatrix}^\dagger i \begin{pmatrix} \overline{\sigma}^\mu \partial_\mu & & \\ & \ddots & \\ & & \overline{\sigma}^\mu \partial_\mu \end{pmatrix} \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_N \end{pmatrix} + \frac{im}{2}(\begin{pmatrix} \chi_1 \\ \vdots \\ \chi_N \end{pmatrix}^T \begin{pmatrix} \sigma^2 & & \\ & \ddots & \\ & & \sigma^2 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_N \end{pmatrix} - \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_N \end{pmatrix}^\dagger \begin{pmatrix} \sigma^2 & & \\ & \ddots & \\ & & \sigma^2 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_N \end{pmatrix}^*)]$$

This has $O(N)$ symmetry, since, for

$$O(N) = \begin{pmatrix} n_{11} & ... & n_{1N} \\ \vdots & \ddots & \vdots \\ n_{N1} & ... & n_{NN} \end{pmatrix},$$

we have

$$(O(N)\otimes Id_2)^T \begin{pmatrix} \overline{\sigma}^\mu\partial_\mu & & \\ & \ddots & \\ & & \overline{\sigma}^\mu\partial_\mu \end{pmatrix} (O(N)\otimes Id_2)$$

$$= (O(N)\otimes Id_2)^T \begin{pmatrix} \overline{\sigma}^\mu\partial_\mu \begin{pmatrix} n_{11} & \\ & n_{11} \end{pmatrix} & \overline{\sigma}^\mu\partial_\mu \begin{pmatrix} n_{12} & \\ & n_{12} \end{pmatrix} & \ldots & \overline{\sigma}^\mu\partial_\mu \begin{pmatrix} n_{1N} & \\ & n_{1N} \end{pmatrix} \\ \overline{\sigma}^\mu\partial_\mu \begin{pmatrix} n_{21} & \\ & n_{21} \end{pmatrix} & \overline{\sigma}^\mu\partial_\mu \begin{pmatrix} n_{22} & \\ & n_{22} \end{pmatrix} & \ldots & \overline{\sigma}^\mu\partial_\mu \begin{pmatrix} n_{2N} & \\ & n_{2N} \end{pmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\sigma}^\mu\partial_\mu \begin{pmatrix} n_{N1} & \\ & n_{N1} \end{pmatrix} & \overline{\sigma}^\mu\partial_\mu \begin{pmatrix} n_{N2} & \\ & n_{N2} \end{pmatrix} & \ldots & \overline{\sigma}^\mu\partial_\mu \begin{pmatrix} n_{NN} & \\ & n_{NN} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \overline{\sigma}^\mu\partial_\mu[1] & \overline{\sigma}^\mu\partial_\mu[0] & \ldots & \overline{\sigma}^\mu\partial_\mu[0] \\ \overline{\sigma}^\mu\partial_\mu[0] & \overline{\sigma}^\mu\partial_\mu[1] & \ldots & \overline{\sigma}^\mu\partial_\mu[0] \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\sigma}^\mu\partial_\mu[0] & \overline{\sigma}^\mu\partial_\mu[0] & \ldots & \overline{\sigma}^\mu\partial_\mu[1] \end{pmatrix}$$

The calculation is done the same way for $(O(N)\otimes Id_2)^T \begin{pmatrix} \sigma^2 & & \\ & \ddots & \\ & & \sigma^2 \end{pmatrix} (O(N)\otimes Id_2)$.

5. We take the hint given in Peskin & Schroeder, and examine the quantized Dirac field. We showed in part 3 that a Dirac field can be written in terms of χ, so we combine these two identities:

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \chi_1 \\ i\sigma^2\chi_2^* \end{pmatrix} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s (a^s_{\mathbf{p}}u^s(p)e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}}v^s(p)e^{ip\cdot x})$$

Thus we have

$$\chi_1 = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s (a^s_{\mathbf{p}} \sqrt{p\cdot\sigma}\xi^s e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}} \sqrt{p\cdot\sigma}\eta^s e^{ip\cdot x})$$
$$\chi_2^* = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s (-i\sigma^2 a^s_{\mathbf{p}} \sqrt{p\cdot\overline{\sigma}}\xi^s e^{-ip\cdot x} + i\sigma^2 b^{s\dagger}_{\mathbf{p}} \sqrt{p\cdot\overline{\sigma}}\eta^s e^{ip\cdot x})$$

Taking the previous problem as guidance, we'll let $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. From 3.144 in Peskin & Schroeder, we know that

$$(v^s(p))^* = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\overline{\sigma}}\xi^s \end{pmatrix} \Rightarrow$$
$$-i\sigma^2\sqrt{p\cdot\overline{\sigma}}\xi^s = (\sqrt{p\cdot\sigma}\eta^s)^*,$$
$$i\sigma^2\sqrt{p\cdot\overline{\sigma}}\eta^s = -(\sqrt{p\cdot\sigma}\xi^s)^*$$

We can use this to get rid of the peskiny ηs, and use the identity $\sigma\sigma^2 = \sigma^2(-\sigma^*)$:

$$\chi_1 = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s (a^s_{\mathbf{p}} \sqrt{p\cdot\sigma}\xi^s e^{-ip\cdot x} - b^{s\dagger}_{\mathbf{p}} i\sigma^2 \sqrt{p\cdot\overline{\sigma}^*}\xi^s e^{ip\cdot x})$$
$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s (a^s_{\mathbf{p}} \sqrt{p\cdot\sigma}\xi^s e^{-ip\cdot x} - b^{s\dagger}_{\mathbf{p}} \sqrt{p\cdot\sigma} i\sigma^2 \xi^s e^{ip\cdot x})$$
$$\chi_2 = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s (a^{s*}_{\mathbf{p}} \sqrt{p\cdot\sigma}\eta^s e^{ip\cdot x} - b^{s\dagger *}_{\mathbf{p}} \sqrt{p\cdot\sigma}\xi^s e^{-ip\cdot x})$$
$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s (-a^{s*}_{\mathbf{p}} \sqrt{p\cdot\sigma} i\sigma^2\xi^s e^{ip\cdot x} - b^{s\dagger *}_{\mathbf{p}} \sqrt{p\cdot\sigma}\xi^s e^{-ip\cdot x})$$

Plugging this into our initial Majorana equation, we have

$$ip\cdot\overline{\sigma} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s (a^s_{\mathbf{p}} \sqrt{p\cdot\sigma}\xi^s e^{-ip\cdot x} - b^{s\dagger}_{\mathbf{p}} \sqrt{p\cdot\sigma} i\sigma^2 \xi^s e^{ip\cdot x})$$
$$= im\sigma^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s (a^{s*}_{\mathbf{p}} \sqrt{p\cdot\sigma^*}\xi^s e^{ip\cdot x} - b^{s\dagger *}_{\mathbf{p}} \sqrt{p\cdot\sigma^*} i\sigma^2\xi^s e^{-ip\cdot x})$$
$$= m \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s (a^{s*}_{\mathbf{p}} \sqrt{p\cdot\overline{\sigma}} i\sigma^2\xi^s e^{ip\cdot x} - b^{s\dagger *}_{\mathbf{p}} \sqrt{p\cdot\overline{\sigma}}\xi^s e^{-ip\cdot x})$$

Since $(p\cdot\overline{\sigma})(p\cdot\sigma) = m^2$, the equation is satisfied if and only if $a^{s*}_{\mathbf{p}} = -b^{s\dagger}_{\mathbf{p}}$. The computation for χ_2 follows in the same way. Thus we have

$$\chi_1 = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s (a^s_{\mathbf{p}}\sqrt{p\cdot\sigma}\xi^s e^{-ip\cdot x} + a^{s*}_{\mathbf{p}}\sqrt{p\cdot\sigma}i\sigma^2\xi^s e^{ip\cdot x}),$$
$$\chi_2 = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s (a^s_{\mathbf{p}}\sqrt{p\cdot\sigma}\xi^s e^{-ip\cdot x} - a^{s*}_{\mathbf{p}}\sqrt{p\cdot\sigma}i\sigma^2\xi^s e^{ip\cdot x})$$

To check that these satisfy the canonical anti-commutation relation, first we compute $\{\chi_1, \chi_2^\dagger\}$:

$$\begin{aligned}
\{\chi_1, \chi_2^\dagger\} = &\int \frac{d^3p d^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \\
&\times \sum_{r,s}[\{a^s_{\mathbf{p}}, a^{r\dagger}_{\mathbf{q}}\}\sqrt{p\cdot\sigma}\xi^s e^{-i(p\cdot x - q\cdot y)}\xi^{r\dagger}\sqrt{q\cdot\sigma} \\
&- \{a^s_{\mathbf{p}}, a^{r*\dagger}_{\mathbf{q}}\}\sqrt{p\cdot\sigma}\xi^s e^{-i(p\cdot x + q\cdot y)}\xi^{r\dagger}i\sigma^2\sqrt{q\cdot\sigma} \\
&+ \{a^{s*}_{\mathbf{p}}, a^{r\dagger}_{\mathbf{q}}\}\sqrt{p\cdot\sigma}(i\sigma^2)^T\xi^s e^{i(p\cdot x + q\cdot y)}\xi^{r\dagger}\sqrt{q\cdot\sigma} \\
&- \{a^{s*}_{\mathbf{p}}, a^{r*\dagger}_{\mathbf{q}}\}\sqrt{p\cdot\sigma}(i\sigma^2)^T\xi^s e^{-i(q\cdot x - p\cdot y)}\xi^{r\dagger}i\sigma^2\sqrt{q\cdot\sigma} \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}[E_{\mathbf{p}}e^{-ip\cdot(x-y)} - E_{\mathbf{p}}e^{ip\cdot(x-y)}] \\
&= \frac{1}{2}\delta^{(3)}(\mathbf{x}-\mathbf{y}) - \frac{1}{2}\delta^{(3)}(\mathbf{y}-\mathbf{x}) = 0
\end{aligned}$$

The other commutators are computed in a similar way. We have

$$\begin{aligned}
\{\chi_1, \chi_1^\dagger\} = \{\chi_2, \chi_2^\dagger\} &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}[E_{\mathbf{p}}e^{-ip\cdot(x-y)} + E_{\mathbf{p}}e^{ip\cdot(x-y)}] \\
&= \delta^{(3)}(\mathbf{x}-\mathbf{y})
\end{aligned}$$

With other conjugate-transpose combinations given, these translate to the commutators of the operators in the quantized Dirac field, which all turn to 0. Therefore we have our canonical anticommutation relations:

$$\{\chi_a, \chi_b^\dagger\} = \delta^{(3)}(\mathbf{x}-\mathbf{y})\delta_{ab}$$

The conjugate momentum is $i\chi^\dagger$, so the Hamiltonian is then

$$H = \int d^3x[\chi^\dagger i\sigma^j \partial_j \chi - \frac{im}{2}(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*)]$$

We start with the first term:

$$\begin{aligned}
\int d^3x \chi^\dagger i\sigma^j\partial_j\chi &= \int d^3x\chi^\dagger q_j\sigma^j\chi \\
&= \int d^3x \int \frac{d^3pd^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\times \\
&\sum_{r,s}[a_{\mathbf{p}}^{r\dagger}\xi^{r\dagger}\sqrt{p\cdot\sigma}\sigma^j q_j e^{-ix\cdot(q-p)}a_{\mathbf{q}}^s\sqrt{q\cdot\sigma}\xi^s \\
&+ a_{\mathbf{p}}^{r\dagger}\xi^{r\dagger}\sqrt{p\cdot\sigma}\sigma^j q_j e^{ix\cdot(p+q)}a_{\mathbf{q}}^{s*}\sqrt{q\cdot\sigma}i\sigma^2\xi^s \\
&+ a_{\mathbf{p}}^{r*\dagger}\xi^{r\dagger}(i\sigma^2)^T\sqrt{p\cdot\sigma}\sigma^j q_j e^{-ix\cdot(p+q)}a_{\mathbf{q}}^s\sqrt{q\cdot\sigma}\xi^s \\
&+ a_{\mathbf{p}}^{r*\dagger}\xi^{r\dagger}(i\sigma^2)^T\sqrt{p\cdot\sigma}\sigma^j q_j e^{-ix\cdot(p-q)}a_{\mathbf{q}}^{s*}\sqrt{q\cdot\sigma}i\sigma^2\xi^s] \\
&= \int \frac{d^3pd^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\times \\
&\sum_{r,s}[a_{\mathbf{p}}^{r\dagger}\xi^{r\dagger}\sqrt{p\cdot\sigma}\sigma^j q_j\delta^{(3)}(q-p)a_{\mathbf{q}}^s\sqrt{q\cdot\sigma}\xi^s \\
&+ a_{\mathbf{p}}^{r\dagger}\xi^{r\dagger}\sqrt{p\cdot\sigma}\sigma^j q_j\delta^{(3)}(p+q)a_{\mathbf{q}}^{s*}\sqrt{q\cdot\sigma}i\sigma^2\xi^s \\
&+ a_{\mathbf{p}}^{r*\dagger}\xi^{r\dagger}(i\sigma^2)^T\sqrt{p\cdot\sigma}\sigma^j q_j\delta^{(3)}(p+q)a_{\mathbf{q}}^s\sqrt{q\cdot\sigma}\xi^s \\
&+ a_{\mathbf{p}}^{r*\dagger}\xi^{r\dagger}(i\sigma^2)^T\sqrt{p\cdot\sigma}\sigma^j q_j\delta^{(3)}(p-q)a_{\mathbf{q}}^{s*}\sqrt{q\cdot\sigma}i\sigma^2\xi^s] \\
&= \int\frac{d^3p}{2E_{\mathbf{p}}}\sum_{r,s}[a_{\mathbf{p}}^{r\dagger}\xi^{r\dagger}\sqrt{p\cdot\sigma}p_j\sigma^j a_{\mathbf{p}}^s\sqrt{p\cdot\sigma}\xi^s \\
&+ a_{\mathbf{p}}^{r\dagger}\xi^{r\dagger}\sqrt{p\cdot\sigma}(-p_j)\sigma^j a_{\mathbf{-p}}^{s*}\sqrt{-p\cdot\sigma}i\sigma^2\xi^s \\
&+ a_{\mathbf{p}}^{r*\dagger}\xi^{r\dagger}(i\sigma^2)^T\sqrt{p\cdot\sigma}(-p_j)\sigma^j a_{\mathbf{-p}}^s\sqrt{-p\cdot\sigma}\xi^s \\
&+ a_{\mathbf{p}}^{r*\dagger}\xi^{r\dagger}(i\sigma^2)^T\sqrt{p\cdot\sigma}p_j\sigma^j a_{\mathbf{p}}^{s*}\sqrt{p\cdot\sigma}i\sigma^2\xi^s]
\end{aligned}$$

Notice that $p_j\sigma^j = \frac{1}{2}(p\cdot\sigma - p\cdot\overline{\sigma})$, so this becomes

$$= \int \frac{d^3p}{4E_{\mathbf{p}}} \sum_{r,s} [a_{\mathbf{p}}^{r\dagger}\xi^{r\dagger}((p\cdot\sigma)^2 - m^2)\xi^s a_{\mathbf{p}}^s$$

$$+ a_{\mathbf{p}}^{r\dagger}\xi^{r\dagger}((p\cdot\sigma)^2 - m^2)i\sigma^2\xi^s a_{\text{-}\mathbf{p}}^{s*}$$

$$+ a_{\mathbf{p}}^{r*\dagger}\xi^{r\dagger}(i\sigma^2)^T((p\cdot\sigma)^2 - m^2)\xi^s a_{\text{-}\mathbf{p}}^{s}$$

$$+ a_{\mathbf{p}}^{r*\dagger}\xi^{r\dagger}(i\sigma^2)^T((p\cdot\sigma)^2 - m^2)i\sigma^2\xi^s a_{\mathbf{p}}^{s*}]$$

Using the identity $(p\cdot\sigma)^2 = E_{\mathbf{p}}^2 + |\mathbf{p}|^2$, we have $(p\cdot\sigma)^2 - m^2 = |\mathbf{p}|^2$, so we have

$$\int d^3x \chi^\dagger i\sigma^j\partial_j\chi = \int d^3p \frac{1}{2E_{\mathbf{p}}} \sum_s [a_{\mathbf{p}}^{s\dagger}|\mathbf{p}|^2 a_{\mathbf{p}}^s - a_{\mathbf{p}}^{sT}|\mathbf{p}|^2 a_{\mathbf{p}}^{s*}]$$

We now check out $\int d^3x \frac{m}{2}(\chi^T i\sigma^2\chi)$:

$$\int d^3x \frac{m}{2}(\chi^T i\sigma^2\chi) = \int d^3x \frac{m}{2}\int \frac{d^3pd^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\times$$

$$= \sum_{s,r}[a_{\mathbf{p}}^{sT}\xi^{sT}i\sigma^2\sqrt{p\cdot\overline{\sigma}}\sqrt{q\cdot\sigma}\xi^r a_{\mathbf{q}}^r e^{-i(p+q)\cdot x}$$

$$+ a_{\mathbf{p}}^{sT}\xi^{sT}i\sigma^2\sqrt{p\cdot\overline{\sigma}}\sqrt{q\cdot\sigma}i\sigma^2\xi^r a_{\mathbf{q}}^{r*} e^{-i(p-q)\cdot x}$$

$$+ a_{\mathbf{p}}^{s*T}\xi^{sT}(i\sigma^2)^T i\sigma^2\sqrt{p\cdot\overline{\sigma}}\sqrt{q\cdot\sigma}\xi^r a_{\mathbf{q}}^r e^{-i(q-p)\cdot x}$$

$$+ a_{\mathbf{p}}^{s*T}\xi^{sT}(i\sigma^2)^T i\sigma^2\sqrt{p\cdot\overline{\sigma}}\sqrt{q\cdot\sigma}i\sigma^2\xi^r a_{\mathbf{q}}^{r*} e^{i(q+p)\cdot x}]$$

$$= \frac{m}{2}\int d^3p \frac{1}{2E_{\mathbf{p}}}\times$$

$$\sum_{s,r}[a_{\mathbf{p}}^{sT}\xi^{sT}i\sigma^2\sqrt{p\cdot\overline{\sigma}}\sqrt{p\cdot\sigma}\xi^r a_{\mathbf{p}}^r$$

$$+ a_{\mathbf{p}}^{sT}\xi^{sT}i\sigma^2\sqrt{p\cdot\overline{\sigma}}\sqrt{p\cdot\sigma}i\sigma^2\xi^r a_{\mathbf{p}}^{r*}$$

$$+ a_{\mathbf{p}}^{s*T}\xi^{sT}(i\sigma^2)^T i\sigma^2\sqrt{p\cdot\overline{\sigma}}\sqrt{p\cdot\sigma}\xi^r a_{\mathbf{p}}^r$$

$$+ a_{\mathbf{p}}^{s*T}\xi^{sT}(i\sigma^2)^T i\sigma^2\sqrt{p\cdot\overline{\sigma}}\sqrt{p\cdot\sigma}i\sigma^2\xi^r a_{\mathbf{p}}^{r*}]$$

$$= \frac{m}{2}\int d^3p \frac{1}{2E_{\mathbf{p}}}\sum_s[-a_{\mathbf{p}}^{sT}ma_{\mathbf{p}}^{s*} + a_{\mathbf{p}}^{s*T}m^2a_{\mathbf{p}}^s]$$

Since $\chi^\dagger i\sigma^2\chi^* = (\chi^T i\sigma^2\chi)^*$, we know that $\frac{im}{2}\chi^\dagger\sigma^2\chi^*$ is $-\frac{im}{2}\chi^T\sigma^2\chi$. Thus the entire $\int d^3x \frac{im}{2}(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*)$ term is $\int d^3p\frac{1}{2E_{\mathbf{p}}}\sum_s[a_{\mathbf{p}}^{s*T}m^2a_{\mathbf{p}}^s - a_{\mathbf{p}}^{sT}m^2a_{\mathbf{p}}^{s*}]$.

Combining this with the first term, we have

$$H = \int d^3p \frac{1}{2E_{\mathbf{p}}} \sum_s [a_{\mathbf{p}}^{s*T}(|\mathbf{p}|^2 + m^2)a_{\mathbf{p}}^s - a_{\mathbf{p}}^{sT}(|\mathbf{p}|^2 + m^2)a_{\mathbf{p}}^{s*}]$$
$$= \int d^3p \frac{1}{2} \sum_s [a_{\mathbf{p}}^{s*T} a_{\mathbf{p}}^s - a_{\mathbf{p}}^{sT} a_{\mathbf{p}}^{s*}]$$

□