Ballinger's Concordance Invariant from Khovanov Homology

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1 Introduction

Recall that the **orientable slice genus** of a smooth knot $K \subset S^3$ is the least integer g such that $K = \partial\Sigma \hookrightarrow B^4$, where Σ is a smoothly and properly embedded orientable surface in B^4 of genus g. There are many ways to bound the orientable slice genus of a given knot. One famous bound is via the Rasmussen invariant: Lee perturbs the differential on Khovanov homology to get a filtered chain complex, and the degree of a generator yields Rasmussen's s invariant. This invariant of a knot gives the lower bound on the slice genus

$$|s(K)| \leq 2g_s(K) \tag{1}$$

Unfortunately, such invariants generalize to nonorientable genus.

Definition 1. *Let K be a knot. Let F be a connected, non-orientable surface in B^4 with $\partial F = K$. Let $b_1(F) = dim(H_1(F, \mathbb{Q}))$ be the first Betti number of F. The* ***non-orientable slice genus*** *$\gamma_4(K)$ of a knot $K \subset S^3$ is the minimum $b_1(F)$ over all such F.*

This is an important inariant in its own right, as many knots bound non-orientable surfaces.

Example 1. *The torus knots $T_{2,n}$ for n odd. See Figure 1. Rasmussen's invariant is bounded on this knot.*

Non-orientable surfaces have some additional data attached to them, via the normal Euler number $e(F)$.

Definition 2. *The* ***normal Euler number*** $e(F)$ *for a possibly non-orientable surface* F *is the self-intersection number of the zero section. For concretely, take a small isotopic displacement of* $\tilde{F}$ *in the normal direction of* F *and count* $F \cap \tilde{F}$*. The sign of an intersection point* $p \in F \cap \tilde{F}$ *is positive if* $(e_1, e_2, \tilde{e}_1, \tilde{e}_2$ *is positively oriented, for* e_i *an arbitrary basis of* TF *and* $\tilde{e}_i$ *their images.*

There are numerous bounds on the non-orientable slice genus using $e(F)$ using the signature of a knot, Heegaard Floer homology, etc.

Before introducing the specific topic of this paper, we need one more definition:

Definition 3. *Two knots* K, K' *are called* ***concordant*** *if* $K \# -K'$ *is slice. Equivalently, they are concordant if* $\partial(S^1 \times [0,1]) = K \sqcup K'$*. This is the same idea as cobordisms for manifolds.*

The topic of this paper is a concordance invariant $t(k)$ introduced by Ballinger ([1]) coming from Khovanov homology that gives a bound on the non-orientable slice genus. Specifically, the following theorem:

Theorem 1. *(Ballinger) For* K *a knot in* S^3,

$$|t(K) + \sigma(K)| \leq 2\gamma_4(K) \tag{2}$$

2 $\mathcal{F}_3$ Khovanov Homology

Recall that, in the proof that Khovanov Homology Kh was link invariant was a **Frobenius algebra**. We had: a unit $1 \in V$ for m, $\epsilon : V \to \mathbb{Z}$ a counit for Δ, with

$$\epsilon(V_+) = 0 \tag{3}$$

$$\epsilon(V_-) = 1 \tag{4}$$

so for:

1. m is a commutative, associative multiplication,
2. Δ a cocommutative, coassociative comultiplication, and
3. the Frobenius law

$$\Delta \circ m = (m \otimes 1) \cdot (1 \otimes \Delta) \tag{5}$$

Definition 4. *Recall that this is called a* ***Frobenius Algebra****.*

Thus we can have different types of Khovanov homologies for different Frobenius algebras. The particular Frobenius algebra that is useful for defining $t(K)$ is the $\mathcal{F}_3$ Frobenius algebra, instead of the traditional 2-dimensional version. In computing the $\mathcal{F}_3$ Khovanov homology for a knot K, we do the following:

1. Create the cube of resolutions for K
2. Suppose a particular resolution has n circles. Associate to this resolution the module $\mathcal{A}_n = \mathbb{Z}[x_1, ..., x_n]/(x_i^2 = x_j^2)$. Repeat for each resolution.
3. Take the edge maps to be:
   - If two circles merge, identify their polynomial variables: $\mathcal{A}_n \to \mathcal{A}_n/(x_i = x_j) \cong \mathcal{A}_n$
   - If one circle splits, $\mathcal{A}_{n-1} \cong \mathcal{A}_n/(x_i = x_j) \xrightarrow{x_i + x_j} \mathcal{A}_n$

   These build our chain complex, the homology groups of which is the Khovanov homology of the knot we started with.

Remark 1. *In $\mathcal{F}_3$ Khovanov homology, we can recover other flavors of Khovanov homology: take a point p on the knot. Let x_p act on the polynomial ring by the variable of the circle containing p at each vertex. Quotienting by x_p^2 gives standard Khovanov homology, quotienting by $x_p^2 - 1$ gives Lee homology, etc.*

3 The Matrix Factorization Picture

For every resolution of the knot, we want to write down a chain complex with homology $\mathcal{A}_n$. In order to do this, we remember a little bit more about the original knot than what the resolution gives us: add a dotted arc where we resolved the crossings for each resolution. See Figure 2. We can break up this new picture of the knot into these resolved crossing regions, and the rest of the knot is strands connecting them together. This leads us to the notion of a decorated tangle diagram:

Definition 5. *For a tangle diagram D_0, a* ***decorated tangle diagram*** *is a collection of D_0 and any number of dotted arcs meeting D_0 only on their boundary, with the requirement that the union of these and D_0 is a connected diagram. Label each region in the complement of this union, and label them with $x_1, ..., x_n$. Near each arc and each crossing (of D_0), mark one of the four adjacent edges in D_0. See Figure 3*

We get a polynomial ring in these variables. $\mathbb{Z}[x_1, ..., x_n]$. We can now introduce the algebraic framework of this theory:

Definition 6. *Let R be a bigraded ring and $w \in R$ a homogeneous element of degree $(2k, 0)$, for k odd. A **matrix factorization** of w is a bigraded R-module C with an R-linear map $d : C \to C$ that is homogeneous of degree $(k, 0)$, such that $d^2 = w$.*

Remark 2. *The reason we make k be odd is to avoid adding a $\mathbb{Z}_2$ grading to track signs.*

Definition 7. *Given (R, w) as above, a **multifactorization** of w is a bigraded R-module C and a sequence of R-linear homogeneous maps $d_i : C \to C$ of degree (k, i) for $i \geq 0$, such that the sum $D = \sum_{i=0}^{\infty} d_i$ satisfies $D^2 = w$. This multifactorization is denoted (C, D).*

$$\begin{tikzcd}
C_{n,m} \arrow[d, "d_0"] \arrow[dr, "d_1"] \arrow[drr, "d_2"] \arrow[drrrr, "d_i"] & & & & & \\
C_{n+k,m} & C_{n+k,m+1} & C_{n+k,m+2} & \cdots & C_{n+k,m+i} & \cdots
\end{tikzcd}$$

For (C, D) a multifactorization, we notice that (C, d_0) is a matrix factorization called the *vertical factorization*; see the vertical d_0 map in the diagram above.

Given two multifactorizations, we can define a chain map between them:

Definition 8. *A **chain map between two multifactorizations** (C, D) and (C', D') is a map F that is a sequence of homogeneous R-linear maps $f_i : C \to C'$ of degree $(0, i)$ for $i \geq 0$, such that $F = \sum_{i=0}^{\infty} f_i$ satisfies $FD = D'F$.*

Definition 9. *Given a chain map $F : (C, D) \to (C', D')$, a **cone** is the complex*

$$C(F) = C(D)[1] \oplus C'(D') := ... \to C(D)^n \oplus C'(D')^{n-1} \to C(D)^{n+1} \oplus C'(D')^n \to ... \tag{6}$$

We also introduce a useful factorization:

Definition 10. *For a bigraded ring R and two homogeneous elements $a, b \in R$ of degree $2i, 2j$, respectively, such that $ab = w$, a length-1 **Koszul factorization** $K(a, b)$ is a matrix factorization given by $R\{i-j, 0\} \xrightarrow{a} R \xrightarrow{b} R\{i-j, 0\}$. For generally, given sequences $\boldsymbol{a} = a_1, ..., a_n$, $\boldsymbol{b} = b_1, ..., b_n$, a length-$n$ Koszul factorization is given by $K(\boldsymbol{a}, \boldsymbol{b}) = \otimes_{i=1}^{n} K(a_i, b_i)$. Note that these are vertical factorizations.*

We now have the tools to construct our complexes for our knot in this picture. We need a ring associated to our knot. Let $R(D)$ be the subring generated by all differences $x_i - x_j$. If e is an oriented edge in D_0, label the difference $x_l - x_r$ x_e, where x_l and x_r are the labels of the regions to the left and right of e, respectively. Now we need a w. Define the w by $\frac{1}{3}\sum_{e\in\partial D} x_e^3$. Notice that $\sum_{e\in\partial D} x_e = 0$, so the $\frac{1}{3}$ factor allows for simplicity while maintaining integer coefficients for w.

Definition 11. *The multifactorization $C(D)$ is the matrix factorization of $(R(D), w)$, with $w = \frac{1}{3}\sum_{e\in\partial D} x_e^3$.*

The four tangle diagrams shown in Figure 3 are a good place to start calculating $C(D)$. $R(D)$ and w are the same for each one:

$$
\begin{aligned}
R(D) &= \mathbb{Z}[x_0 - x_1, x_0 - x_2, ..., x_2 - x_3], && (7)\\
w &= \frac{1}{3}[(x_0 - x_1)^3 + (x_1 - x_2)^3 + (x_2 - x_3)^3 + (x_3 - x_0)^3] && (8)\\
&= x_0x_1^2 - x_0^2x_1 + x_1x_2^2 - x_1^2x_2 + x_2x_3^2 - x_2^2x_3 + x_3x_0^2 - x_3^2x_0 && (9)\\
&= (x_0 - x_1)(x_1^2 - x_3^2) - (x_0^2 - x_2^2)(x_1 - x_3) && (10)\\
&= (x_0 - x_2)(x_1 - x_3)(x_1 - x_2 + x_3 - x_0) && (11)
\end{aligned}
$$

For D_0, D_1 in Figure 3, $C(D_0)$ and $C(D_1)$ are Koszul factorizations $K(x_0 - x_2, (x_1 - x_3)(x_1 - x_2 + x_3 - x_0))$ and $K(x_1 - x_3, (x_0 - x_2)(x_1 - x_2 + x_3 - x_0))$, respectively, with $C(D_+)$ and $C(D_-)$ the mapping cones on chain maps between $C(D_0)$ and $C(D_1)$.

Definition 12. *For a general decorated tangle diagram D, an **elementary subdiagram** of D is a subdiagram D_e containing at most one dotted arc or crossing. Modulo relabeling variables, every elementary subdiagram is equivalent to one of D_0, D_1, D_+, D_-.*

Using this fact, we can define the *total chain complex* of a decorated tangle diagram by

$$C(D) = \otimes_{D_e \subset D \text{ elementary}} C(D_e) \otimes_{R(D_e)} R(D) \qquad (12)$$

For a knot, this is how we package local matrix factorizations for the individual resolutions into one double complex that computes the Khovanov homology.

Remark 3. *So far we have neglected the use of the marked edges in each arc and crossing. This is because the marked edge only defines a sign in the differential in $C(D)$. If two diagrams differ only in marked edges, a diagonal matrix is ± 1s on the diagonal corresponding to the differences defines an isomorphism between the associated Koszul factorizations.*

Remark 4. *The reason we've only been considering tangles so far is because this allows us to consider crossings/resolutions without worrying about the connecting strands between these crossings/resolutions; one may wonder why the differential on the $C(D)$ complexes squares to a generally nonzero element w. When we apply this tensor product, all of the constants cancel out and we get an honest chain complex.*

The edge maps in the Khovanov complex (maps from one resolution to the other at a crossing site) come from matrix factorization homomorphisms, which we will call **saddle maps** (see the bottom of Figure 4). This is completely local, which allows us to prove Reidemeister invariance on the tangle level. Thus we build a double complex $C(K)$ out of the matrix factorizations of all resolutions in K:

- The vertical differentials are the matrix factorization differentials
- The horizontal differentials are saddle maps.

Proposition 1. *$C(K)$ is a 1-homotopy invariant of K.*

The idea of a proof of this is twofold: moving dotted arcs around each other in certain ways and switching dotted arcs retains the factorization $C(K)$ (See Figure 4), and invariance of Reidemeister moves.

Proposition 2. *If D has no boundary and n closed components, then the homology of $C(D)$ is isomorphic to $\mathbb{Z}[x_1, ..., x_n]/(x_i^2 - x_j^2)\{n-1\}$, where x_i are the variables associated to each edge.*

The idea of the proof of this is to move the dotted arcs in a way that retains $C(D)$ to form a sequence of closed components $K_1, K_2, ..., K_n$ where the dotted arcs connect K_i to K_{i+1}. This allows for $R(D) = \mathbb{Z}[x_1, ..., x_n]$ where x_i is associated to an each on K_i. Calculating $C(D)$ yields the Koszul factorization

$$K((0, ..., 0), (x_1^2 - x_2^2, ..., x_{n-1}^2 - x_n^2)) \tag{13}$$

The homology of this complex is then $R(D)/(x_i^2 - x_j^2)$. Since x_i^2 are in the second part of the Koszul factorization, we need to shift the grading.

Remark 5. *This is the module assigned to n circles by $\mathcal{F}_3$.*

$C(K)$ has a grading $q - 3h$, where q is the q-grading and h is the homological grading. All of the differentials lower this grading by 3.

Remark 6. *The homological grading h is just a filtration, because some differentials increase it by 1, and others don't change it at all.*

Definition 13. *When K is a knot, $C(K)$ is a complex over $\mathbb{Z}[x]$. Thus we can form a* ***reduced complex associated to*** *K, namely $\overline{C(K)} = C(K)/x$. We treat $C(K)$ and $\overline{C(K)}$ as chain complexes with a filtration. The unfiltered complexes are the* ***total complexes****, with homology the* ***total homology****.*

Suppose two decorated tangle diagrams D, D' are the same except for some region, where D has D_0 in this region and D' has D_1. Then $R(D) = R(D')$ and there is a complex $C_{outside}$ such that

$$C(D) = C_{outside} \otimes_{R(D)} C(D_0) \tag{14}$$

$$C(D') = C_{outside} \otimes_{R(D)} C(D_1) \tag{15}$$

The saddle map here $s_{D \to D'} : C(D) \to C(D')\{1\}$ is the tensor product of the elementary saddle map $s_{D_0 \to D_1}$ with the identity on $C_{outside}$. In addition, it is not hard to prove that $C(K \sqcup U) \cong C(K)\{-1\} \oplus C(K)\{1\}$. With the above saddle map, the inclusion map $b_K : C(K) \to C(K \sqcup U)\{-1\}$, and the 2-handle projection map $d_K : C(K \sqcup U) \to C(K)\{-1\}$, we have a map

$$f_\Sigma : C(K) \to C(K')\{-\chi(\Sigma)\} \tag{16}$$

for any cobordism Σ from K to K' with a handle decomposition.

Proposition 3. *If K, K' are knots and Σ has genus g, the map induced by f_Σ on total homology is multiplication by $(2x)^g$. In particular, when Σ is a concordance, f_Σ induces an isomorphism on homology.*

The proof of this uses the fact that the map induced by f_Σ on total homology is equal up to sign to the cobordism map from the $\mathcal{F}_3$ Frobenius extension.

4 The Invariant $t(K)$

Definition 14. *Let k be a field. $t(K)$ is the largest n for which there is a cycle in $\overline{C}(K) \otimes_{\mathbb{Z}} k$, written as a sum of homogeneous elements with filtration grading at least n, generating the total homology.*

Definition 15. *For $i \geq 0$, $T_i(K)$ is the largest n for which there is a cycle in $C(K)$ representing the element $(2x)^i$ in the total homology that can be written as a sum of homogeneous elements with fibration grading at least n.*

Proposition 4. *If K, K' are concordant, then $t(K) \cong t(K')$.*

Proof. The idea is to use f_Σ for an oriented cobordism Σ from K to K'. If g is the genus of Σ, the induced map $f_\Sigma : C(K) \to C(K')$ is filtered and acts on the total homology by $(2x)^g$. This sends a cycle γ representing $(2x)^i$ in the total homology of $C(K)$ supported in filtration degrees greater than or equal to $T_i(K)$ to a cycle $f_\Sigma(\gamma)$ representing $(2x)^{i+g}$ supported in filtration degrees greater than or equal to $T_i(K')$. If $g = 0$, K and K' are concordant. Consider the same bound from the concordance $\tilde{\Sigma} : K' \to K$. This means that $T_i(K) = T_i(K')$, and thus $f_\Sigma, f_{\tilde{\Sigma}}$ are filtered isomorphisms $\overline{C}(K) \otimes k \cong \overline{C}(K') \otimes k$, and thus $t(K) = t(K')$. □

Theorem 2. *(Ballinger) $|t(K) + \sigma(K)| \leq 2\gamma_4(K)$*

The idea of this proof is to generalize to a possibly non-orientable cobordism F from knot K to K'. If such a cobordism F exists with first Betti number $b + 1$ and normal Euler number e, then we have the bound $|t(K) - t(K') - e/2| \leq b$. This is done by factoring F as a concordance, then b non-orientable bands of total Euler number e, then another concordance. Since t is concordance-invariant, we prove the bound when K, K' are related by a sequence of b non-orientable bands. We can do this by just proving this for $b = 1$ and applying the bands

References

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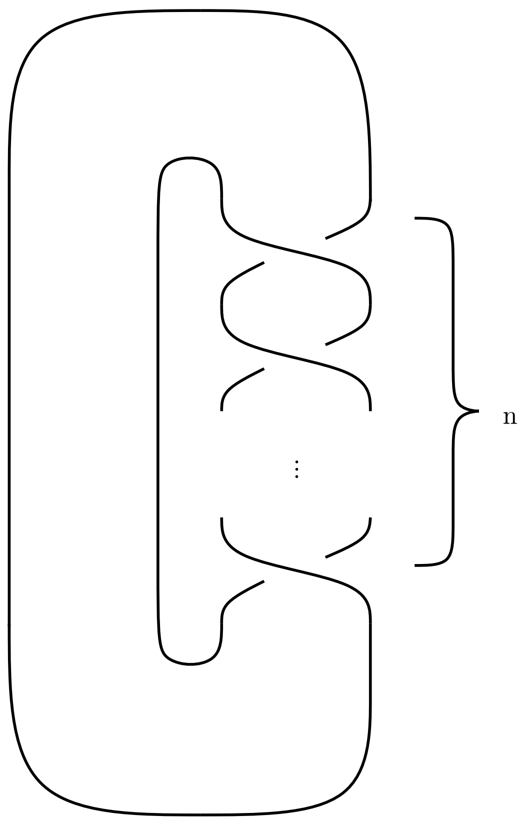

Figure 1: $T_{2,n}$ are knots that bound non-orientable surfaces.

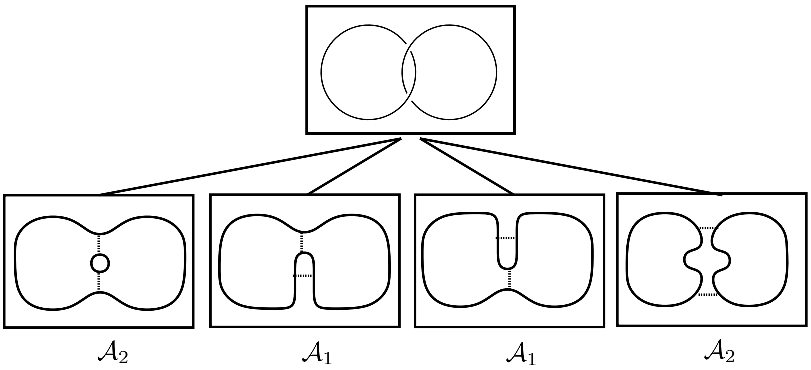

Figure 2: Adding dotted arcs to give a chain complex.

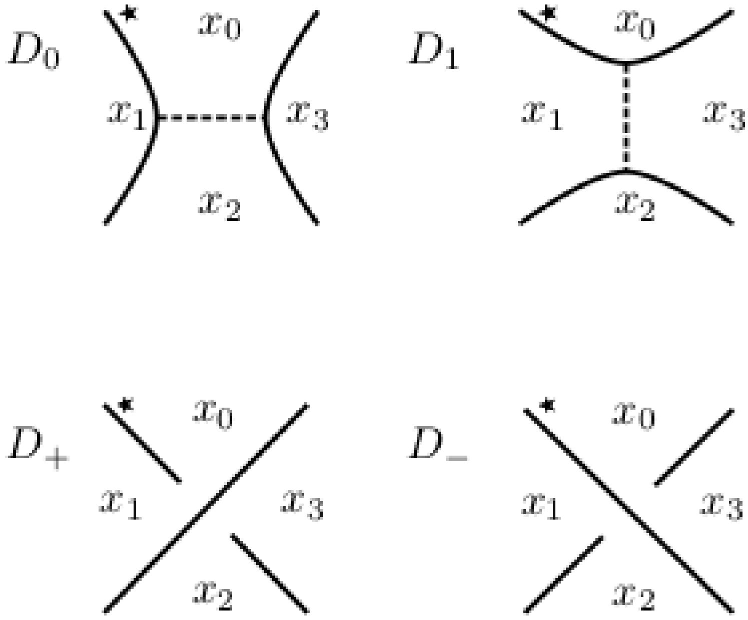

Figure 3: The four basic tangles with mark $\star$. Figure taken from [1].

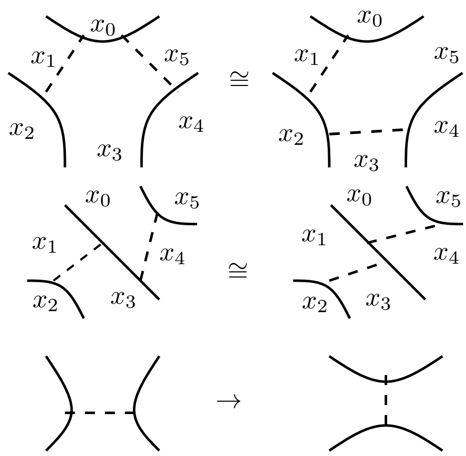

Figure 4: From the top: moving dotted arcs around each other, switching dotted arcs, and a saddle map for a resolution.