Quantum Chaotic Behavior in Black Holes

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I. THE SACHDEV-YE-KITAEV MODEL

The Sachdev-Ye-Kitaev model (SYK) is written in terms of N Majorana fermions Ψ_i, for N even. The **Sachdev-Ye-Kitaev model** is defined by the Hamiltonian

$$H = \frac{1}{4!}\sum_{a,b,c,d} J_{abcd}\psi_a\psi_b\psi_c\psi_d = \sum_{a<b<c<d} J_{abcd}\psi_a\psi_b\psi_c\psi_d \tag{1}$$

for ψ Majorana fermions and J_{abcd} the anti-symmetric coupling tensor, and each independent element $J_{....}$ is a random real number chosen from a Gaussian distribution with average $\langle J_{....}\rangle = 0$ and variance dependent on parameter J: $\langle J^2_{....}\rangle = \frac{3!J^2}{N^3}$. The Hilbert space dimension is $L = 2^{N/2}$. In the limit $N \to \infty$, the model can be solved for any $\beta J := \lambda$. The SYK model has an emergent conformal symmetry, making it a "nearly" 1-dimensional conformal field theory (CFT_1) and is thus is a toy model for "nearly" AdS/"nearly" CFT ($NAdS_2/NCFT_1$), a version of traditional AdS/CFT where we relate quantum-mechanical systems with quantum gravity systems in certain limits. This model can be solved at large N and large βJ in the $\frac{1}{N}$ expansion. We define the **spectral form factor**:

$$g(t) := \frac{\langle Z(\beta+it)Z^*(\beta-it)\rangle_J}{\langle Z(\beta)\rangle_J^2} \tag{2}$$

and study its behavior, shown in Figure 1.

In the region before the dip time, we can understand this curve gravitationally in the $\frac{1}{N}$ expansion. In the region after the dip time, we can understand this curve in the $\frac{1}{L}$ expansion using random matrix theory (RMT). In this report, we discuss the both expansions. In the former region, because $g(t)$ is self-averaging, rather than thinking about the average of the square, we can just think about the average $\langle Z(\beta+it)\rangle_J$. The computation of this quantity is very involved, and we sketch this calculation; in the report, we provide explicit computations for several key steps.

II. $NAdS_2/NCFT_1$ CORRESPONDENCE FOR SYK

We rewrite the disorder average of the spectral form factor in terms of bilocal antisymmetric fields, integrate out the Gaussian couplings J_{ijkl}, integrate out the fermions via a lagrange multiplier field, and consider the saddle point of the resulting action. By introducing and integrating out fluctuations in these bilocal fields, we get an exactly-solvable symplectic integral with action exactly that of Dilaton gravity.

$$\langle Z(\beta+it)\rangle_J = \int_{Diff(S^1)/SL(2,\mathbb{R})} d\Omega \exp\{\frac{N}{(\beta+it)J}\int Sch(\phi,\tau)d\tau\} \tag{3}$$

where Sch is the Schwartzian derivative used in Dilaton gravity.

III. ANALYTIC RAMP FOR A GUE RANDOM MATRIX HAMILTONIAN

If we compute $\langle Z\rangle$ for an $(L\times L)$ GUE random matrix as our Hamiltonian, the curve to the right of the dip time looks quite similar to the same region in Figure 1. This is made explicit in the report.

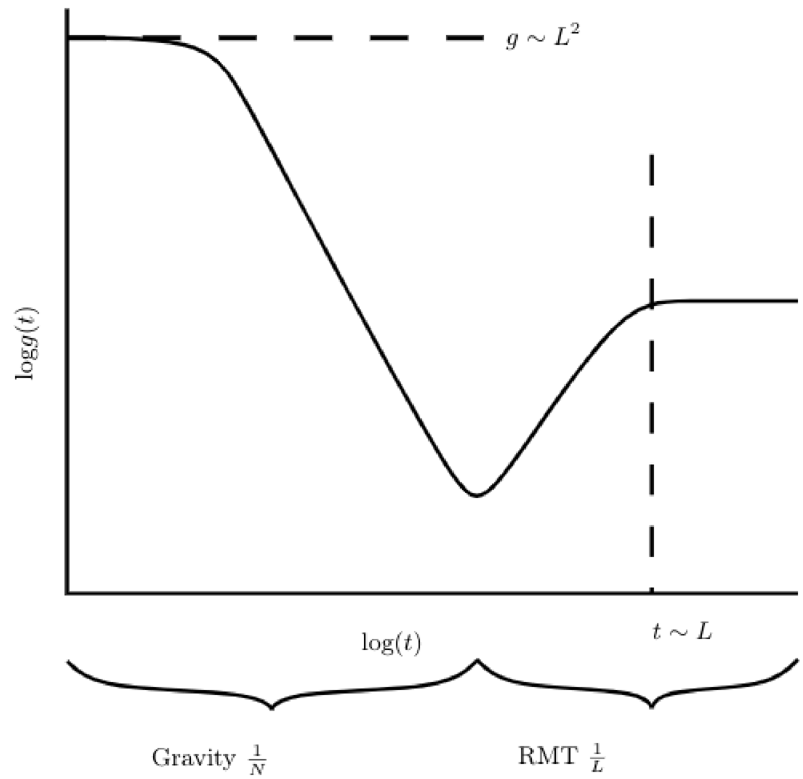

FIG. 1: log-log plot of $g(t)$ for the SYK model. The time of $min[g(t)]$ is known as the "dip time," and the two time regions before and after this dip time are effectively described by different effective theories, with respective order perturbations. Source: A. Lau.

IV. ANALYTIC RAMP FOR SYK

We then analytically derive the ramp behavior of the SYK model, starting with the general definition of ZZ^*, taking GUE statistics and using the effective Schwartzian derivative theory to find a numerical solution that looks fairly close to the exact solution mentioned (which exists). We derive this in the report.

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What are the quantum dynamics of black holes? Does such a system follow quantum-chaotic behavior at late times? In this report we use the **Sachdev-Ye-Kitaev Model** (SYK) to give some guidance for these questions. The SYK model is a frequently studied model in physics due to its relationship with black holes. Here we illustrate what the SYK model is and why it can be used to model black holes at late times. We then characterize its behavior in certain limits and conclude that the late time behavior of this model, and therefore late time behavior of horizon fluctuations in large anti-de Sitter (AdS) black holes, are governed by random matrix dynamics, a hallmark of a quantum chaotic system. The main tool we'll use for this argument is the **spectral form factor** which uses an analytically continued partition function in order to establish this connection.

V. THE SACHDEV-YE-KITAEV MODEL

The Sachdev-Ye-Kitaev model (SYK) is written in terms of N Majorana fermions Ψ_i, for N even.

Definition 1. ***Majorana fermions*** *satisfy the algebra*

$$\psi_i\psi_j + \psi_j\psi_i = 2\delta_{ij}, i,j \in \{1,...,N\} \tag{4}$$

where N is even. These live in a Hilbert space of dimension $L = 2^{N/2}$. We treat the ψ_is as $L \times L$ representations of this algebra.

Example 1. *For $N = 2$, we have*

$$\psi_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5}$$

Definition 2. *The* ***Sachdev-Ye-Kitaev model*** *is defined by the Hamiltonian*

$$H = \frac{1}{4!}\sum_{a,b,c,d} J_{abcd}\psi_a\psi_b\psi_c\psi_d \tag{6}$$

$$= \sum_{a<b<c<d} J_{abcd}\psi_a\psi_b\psi_c\psi_d \tag{7}$$

for $a, b, c, d \in \{1, ..., N\}$. J_{abcd} is the anti-symmetric coupling tensor, and each independent element $J_{....}$ is a random real number chosen from a Gaussian distribution with average $\langle J_{....}\rangle = 0$ and variance dependent on parameter J: $\langle J^2_{....}\rangle = \frac{3!J^2}{N^3}$. The $\frac{3!}{N^3}$ factor is just for normalization convenience; the important part is the dependence on J.

Remark 1. *The most general form of the Hamiltonian introduces a parameter q, representing the interactions of q of the N fermions at a given time:*

$$H = i^{\frac{q}{2}} \sum_{1\leq i_1\leq ...\leq i_q \leq N} j_{i_1,...,i_q}\psi_{i_1}...\psi_{i_q} \tag{8}$$

where

$$\langle j^2_{i_1,...,i_1}\rangle = \frac{J^2(q-1)!}{N^{q-1}} \tag{9}$$

The numerical factors and factors of N are used to simplify the large N limit, and the i factor is to make the Hamiltonian hermitian when $q \equiv 2 \mod (4)$. Notice that, if we have odd $\frac{q}{2}$, the model is not time-reversal-symmetric. For this reason we consider $q = 4$, as this model represents the dominant interactions at low energy.

We can see that, since there is no spatial dependence, this models a (0+1)-dimensional quantum mechanical system. In the limit $N \to \infty$, the model can be solved for any $\beta J := \lambda$. The Sachdev-Ye-Kitaev model has an emergent conformal symmetry and is thus is a toy model for "nearly" AdS/"nearly" CFT, a version of traditional AdS/CFT where we relate quantum-mechanical systems with quantum gravity systems in certain limits. It is thus a model of a system living in 0-space and 1-time, and it is related to a quantum gravity system in 1-space and 1-time. This model can be solved at large N and large βJ in the $\frac{1}{N}$ expansion. The way to do this is to integrate over the couplings $J_{....}$ and taking the disorder average, that is the average over the ensemble of random couplings. This results in a complicated theory with large parameter N, which allows one to solve it in an expansion in $\frac{1}{N}$. One can show that correlation functions in this model in the $\frac{1}{N}$ expansion agree with correlation functions of fields propagating in a 2-dimensional black hole. If we analytically continue the partition function $Z(\beta) = Tr[e^{-\beta H}]$ to $\beta + it$, we get

$$Z(\beta + it) = Tr[e^{-\beta H - iHt}] \tag{10}$$

$$= \sum_{m=1}^{L} e^{-\beta E_m - iE_m t} \tag{11}$$

If t is small, we are summing positive quantities. But as t grows, we are summing different phases, which give partial cancellations in the sum, so $Z(\beta+it)$ will decrease in time. $Z(\beta + it)$ does not, however, decrease to zero. It will become small but still oscillate, and at very large times it will have recurrences back to its original value.

Remark 2. *In computing the quantity $Z(\beta + it)$ in the 2-dimensional gravity theory, we run into trouble. We believe that black holes are finite-entropy quantum systems, and for finite-entropy quantum systems this quantity should not go to 0. But in the perturbative expansion*

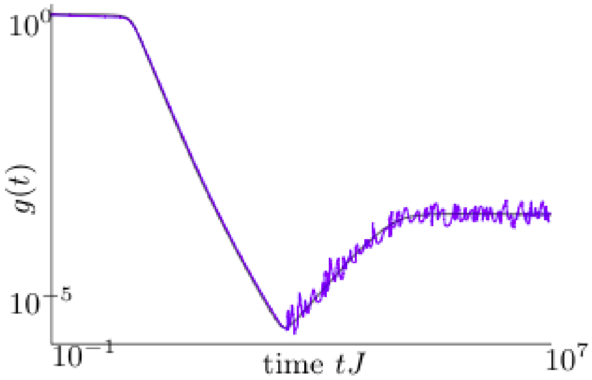

FIG. 2: $g(t)$ for $N = 34, L = 2^{17}, \beta J = 5$. Single and averaged over 90 samples. Source: A. Lau.

in the gravitational theory, it turns out that $Z(\beta + it)$ does go to 0. This is an example of the ***black hole information problem****. As of now this is a major problem in theoretical physics.*

We seek to understand $Z(\beta+it)$ in the SYK model. We want to compute this quantity in the disorder average: $\langle Z \rangle_J$. Here there is a problem, however. We have a continuous spectrum, and this quantity can and does go to zero. We rectify this by noticing that, for a given realization of the model, the typical value of $Z(\beta + it)$ will not be zero.

Definition 3. *Instead we define the* ***spectral form factor:***

$$g(t) := \frac{\langle Z(\beta + it) Z^*(\beta - it)\rangle_J}{\langle Z(\beta)\rangle_J^2} \tag{12}$$

and this should not go to zero.

Cotler et. al. in[1] compute $g(t)$ for $N = 34, L = 2^{17}, \beta J = 5$ using 90 independent samples. The result is a figure like that of Figure 2.

The initial plateau is of order $g(t) \sim L^2$; there are no phases in the sum, and we sum up L terms, and square it to get $g(t)$. The initial drop is referred to as the *slope*, the minimum is referred to as the *dip*, the increase is referred to as the *ramp*, and the plateau is just the *plateau*. At the plateau $g(t) \sim L$, and this occurs at time $t \sim L$. The time of the dip is not known. The graph in blue is the qualitative behavior of a single J. The oscillations are of order the function itself (the function is self-averaging until the dip time).

In the region before the dip time, we can understand this curve in the $\frac{1}{N}$ expansion. In the region after the dip time, we can understand this curve in the $\frac{1}{L}$ expansion. First we discuss the former expansion. In this region, because $g(t)$ is self-averaging, rather than thinking about the average of the square, we can just think about the average $\langle Z(\beta+it)\rangle_J$. The computation of this quantity is very involved; we outline the calculation as in[3], specifying explicit calculations for several steps.

VI. $NAdS_2/NCFT_1$ CORRESPONDENCE FOR SYK

Definition 4. *We define the* ***Euclidean propagator*** *as*

$$G(\tau) = \langle T(\psi(\tau)\psi(0))\rangle \tag{13}$$

$$= \langle\psi(\tau)\psi(0)\rangle\theta(\tau) - \langle\psi(0)\psi(\tau)\rangle\theta(-\tau) \tag{14}$$

and the bilocal function

$$G(\tau_1, \tau_2) = G(\tau_1 - \tau_2) \tag{15}$$

At large N this field is equal to the two-point function of the fermions

$$G(\tau_1, \tau_2) = \frac{1}{N}\sum_{i=1}^{N}\langle\psi_i(\tau_1)\psi_i(\tau_2)\rangle \tag{16}$$

Definition 5. *We define the* ***self energy*** *$\Sigma(\tau, \tau')$ by*

$$\Sigma(\tau, \tau') = \Sigma(\tau - \tau'), \tag{17}$$

$$\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega), \tag{18}$$

$$\Sigma(\tau) = J^2[G(\tau)]^3 \tag{19}$$

At $\beta = \infty$, we take ω to be continuous, and at finite β, we take $\omega = \frac{2\pi}{\beta}(n + \frac{1}{2})$.

At strong coupling, we ignore the first term in (18) to get the approximation

$$\int d\tau' G(\tau, \tau')\Sigma(\tau', \tau'') = -\delta(\tau - \tau') \tag{20}$$

$$\Sigma(\tau, \tau') = J^2[G(\tau, \tau')]^3 \tag{21}$$

Under the reparametrization $\tau \mapsto f(\tau)$, we see that

$$J^2 \int |\frac{df}{d\tau'}| d\tau' G(f(\tau), f(\tau'))G(f(\tau'), f(\tau''))^3 \tag{22}$$

$$= -\delta(f(\tau) - f(\tau'')) = -\frac{1}{|f'(\tau'')|}\delta(\tau - \tau'') \tag{23}$$

Thus this equation is invariant under the reparametrization group $Diff(\mathbb{R})$. The ansatz for G is

$$G_c(\tau) = \frac{b}{\sqrt{|\tau|}} sgn(\tau) \tag{24}$$

for b a parameter determined by inserting this ansatz into the simplified equations. The exact solution of this form has its $Diff(\mathbb{R})$ symmetry spontaneously broken by $SL(2,\mathbb{R})$, and we compactify $\mathbb{R}$ to S^1 by the reparametrizations $f(\tau) \mapsto e^{2\pi i\tau/\beta}$, giving us the symplectic manifold $Diff(S^1)/SL(2,\mathbb{R})$. Such symmetry makes the SYK model a **nearly conformal field theory**, or $NCFT_1$. This gives rise to duality with nearly anti-de Sitter spaces, and thus $NAdS_2/NCFT_1$ duality.

The computation of $\langle Z(\beta + it)\rangle_J$ is very involved, and we give a sketch here of the computation performed by

Maldacena and Stanford in[3], with a few explicit calculations. We compute the disorder average of 2 copies (replicas) of the partition function, labeled by α, β=1,2:

$$\langle Z(\beta+it)Z(\beta-it)\rangle_J \tag{25}$$
$$= \int \mathcal{D}\psi_i^\alpha \mathcal{D}J_{ijkl} \exp\{-a\sum_{ijkl} J_{ijkl}^2\} \tag{26}$$
$$\times \exp\{-\sum_{\alpha=1}^{2}\int d\tau(\frac{1}{2}\sum_i \psi_i^\alpha \frac{d}{d\tau}\psi_i^\alpha \tag{27}$$
$$-\frac{1}{4!}\sum_{ijkl} J_{ijkl}\psi_i^\alpha\psi_j^\alpha\psi_k^\alpha\psi_l^\alpha)\} \tag{28}$$

where α is the replica index and a is an (unimportant in the long run) parameter. Because J_{ijkl} is a Gaussian, we perform the integration

$$\int \mathcal{D}J_{ijkl}\exp\{\sum_\alpha\sum_{ijkl}[\frac{1}{4!}\int d\tau J_{ijkl}\psi_i^\alpha\psi_j^\alpha\psi_k^\alpha\psi_l^\alpha - aJ_{ijkl}^2]\} \tag{29}$$
$$= \exp\{\frac{a4!}{4(4!)^2a^2}\sum_{\alpha\beta}\int d\tau_1 d\tau_2(\sum_{i=1}^{N}\frac{1}{N}\psi_i^\alpha(\tau_1)\psi_i^\beta(\tau_2))^4\} \tag{30}$$
$$= \exp\{\frac{J^2N}{8}\sum_{\alpha\beta}\int d\tau_1 d\tau_2(\sum_{i=1}^{N}\frac{1}{N}\psi_i^\alpha(\tau_1)\psi_i^\beta(\tau_2))^4\} \tag{31}$$

where the last inequality follows from the fact that there are 4! terms for any fixed $ijkl$ and the antisymmetry of J_{ijkl} and the anticommutation of the ψ_i fields. Our expression becomes

$$\langle Z(\beta+it)Z(\beta-it)\rangle_J = \tag{32}$$
$$\int \mathcal{D}\psi_i^\alpha \exp\{-\frac{1}{2}(\sum_{\alpha=1}^{2}\sum_{i=1}^{N}\int d\tau\psi_i^\alpha\frac{d}{d\tau}\psi_i^\alpha \tag{33}$$
$$-\frac{J^2N}{4}\sum_{\alpha\beta}\int d\tau_1 d\tau_2(\sum_{i=1}^{N}\frac{1}{N}\psi_i^\alpha(\tau_1)\psi_i^\beta(\tau_2))^4)\} \tag{34}$$

This gives a bilocal action for the fermions. We can integrate out the fermions if we introduce a field $\tilde{G}^{\alpha\beta}(\tau_1,\tau_2)$ and set it equal to $\frac{1}{N}\sum_{i=1}^{N}\psi_i^\alpha(\tau_1)\psi_i^\beta(\tau_2)$. The motivation for this is the fact that the above contains an $O(N)$ symmetry via $\psi_i\psi^i \mapsto \psi\mathcal{O}\mathcal{O}^T\psi = \psi_i\psi^i$. To introduce this into our path integral, we use the analogous $\delta(x) = \int dk e^{ikx}$:

$$\delta(\tilde{G}^{\alpha\beta}(\tau_1,\tau_2) - \frac{1}{N}\sum_{i=1}^{N}\psi_i^\alpha(\tau_1)\psi_i^\beta(\tau_2)) \propto \tag{35}$$
$$\int d\tilde{\Sigma}^{\alpha\beta}(\tau_1,\tau_2)\exp\{-\frac{N}{2}\sum^{\alpha\beta}(\tau_1,\tau_2)(\tilde{G}^{\alpha\beta}(\tau_1,\tau_2) \tag{36}$$
$$-\frac{1}{N}\sum_{i=1}^{N}\psi_i^\alpha(\tau_1)\psi_i^\beta(\tau_2))\} \tag{37}$$

where $\tilde{\Sigma}^{\alpha\beta}(\tau_1,\tau_2)$ acts as a Lagrange multiplier. Inserting this into our expression yields

$$\langle Z(\beta+it)Z(\beta-it)\rangle_J \tag{38}$$
$$= \int \mathcal{D}\psi_i \mathcal{D}\tilde{G}^{\alpha\beta}\mathcal{D}\tilde{\Sigma}^{\alpha\beta}\exp\{ \tag{39}$$
$$-\sum_{\alpha\beta=1}^{2}\sum_{i=1}^{N}\frac{1}{2}\int d\tau_1 d\tau_2[\psi_i^\alpha(\tau_1) \tag{40}$$
$$\times(\delta^{\alpha\beta}\delta(\tau_1-\tau_2) - \tilde{\Sigma}^{\alpha\beta}(\tau_1,\tau_2))\psi_i^\beta(\tau_2)] \tag{41}$$
$$-\frac{1}{2}\sum_{\alpha\beta}\int d\tau_1 d\tau_2(N\tilde{\Sigma}^{\alpha\beta}(\tau_1,\tau_2)\tilde{G}^{\alpha\beta}(\tau_1,\tau_2) \tag{42}$$
$$-\frac{J^2N}{4}(\tilde{G}^{\alpha\beta}(\tau_1,\tau_2))^4)\} \tag{43}$$

Integrating out the fermions, we have

$$\int \mathcal{D}\psi_i^\alpha \exp\{ \tag{44}$$
$$-\sum_{\alpha\beta=1}^{N}\sum_{i=1}^{N}\frac{1}{2}\int d\tau_1 d\tau_2[\psi_i^\alpha(\tau_1)(\delta^{\alpha\beta}\delta(\tau_1-\tau_2)\partial_\tau \tag{45}$$
$$-\tilde{\Sigma}^{\alpha\beta}(\tau_1,\tau_2))\psi_i^\beta(\tau_2)]\} \tag{46}$$
$$= \exp\{\frac{N}{2}\sum_{\alpha\beta}\log\det(\delta^{\alpha\beta}\partial_\tau - \tilde{\Sigma}^{\alpha\beta})\} \tag{47}$$

yielding our new expression

$$\langle Z(\beta+it)Z(\beta-it)\rangle_J = \tag{48}$$
$$\int \mathcal{D}\tilde{G}^{\alpha\beta}\mathcal{D}\tilde{\Sigma}^{\alpha\beta}\exp\{\frac{N}{2}\sum_{\alpha\beta}\log\det(\delta^{\alpha\beta}\partial_\tau - \tilde{\Sigma}^{\alpha\beta})\} \tag{49}$$
$$\times\exp\{\frac{1}{2}\sum_{\alpha\beta}\int d\tau_1 d\tau_2(N\tilde{\Sigma}^{\alpha\beta}(\tau_1,\tau_2)\tilde{G}^{\alpha\beta}(\tau_1,\tau_2) \tag{50}$$
$$-\frac{J^2N}{4}(\tilde{G}^{\alpha\beta}(\tau_1,\tau_2))^4)\} \tag{51}$$

Next we assume a replica symmetric saddle point $\tilde{G}^{\alpha\beta} = \delta^{\alpha\beta}\tilde{G}$. Plugging this in, we get

$$\langle Z(\beta+it)Z(\beta-it)\rangle_J \tag{52}$$
$$= \int \mathcal{D}\tilde{G}\mathcal{D}\tilde{\Sigma}\exp\{-M(-\frac{N}{2}\log\det(\partial_\tau - \tilde{\Sigma}) \tag{53}$$
$$+\frac{1}{2}\int d\tau_1 d\tau_2(N\tilde{\Sigma}(\tau_1,\tau_2)\tilde{G}(\tau_1,\tau_2 \tag{54}$$
$$-\frac{J^2N}{4}(\tilde{G}(\tau_1,\tau_2))^4)\} \tag{55}$$

Here we define fluctuations σ, g such that

$$\tilde{G} = G + \frac{g}{|G|}, \tilde{\Sigma} = \Sigma + |G|\sigma \tag{56}$$

The measure is invariant $d\tilde{G}d\tilde{\Sigma} = dg d\sigma$. We expand the action to second order in g and σ and using the saddle point equation $G = (\partial_\tau - \Sigma)^{-1}$, we get the action

$$S = \frac{N}{12J^2}\int d\tau_1...d\tau_4\sigma(\tau_1,\tau_2)J^2 3|G(\tau_1-\tau_2)| \quad (57)$$
$$\times G(\tau_1-\tau_3)G(\tau_2-\tau_4)|G(\tau_3-\tau_4)|\sigma(\tau_3,\tau_4) \quad (58)$$
$$+\frac{1}{2}\int d\tau_1 d\tau_2[g(\tau_1,\tau_2)\sigma(\tau_1,\tau_2) - \frac{1}{2}J^2 3g(\tau_1,\tau_2)^2] \quad (59)$$

Integrating out σ, we get

$$S = \frac{J^2 3N}{4}g([|G(\tau_1-\tau_2)| \quad (60)$$
$$\times G(\tau_1-\tau_3)G(\tau_2-\tau_4)|G(\tau_3-\tau_4)|]^{-1}-1)g \quad (61)$$

Now we go to low energies (the low-temperature limit) and use the conformal expressions G_c, Σ_c. By including the leading non-conformal term in the above action, we consider a small reparametrization $\tau \mapsto \tau_\epsilon(\tau)$ and evaluate the action on $\delta_\epsilon G_c$. The result is

$$S = N\frac{\alpha}{J}\int_0^\beta d\tau\frac{1}{2}[(\epsilon'')^2 - (\frac{2\pi}{\beta})^2(\epsilon')^2] \quad (62)$$

where α is a constant. Generalizing to finite reparametrizations $\tau \mapsto \phi(\tau)$, we take the expansion

$$\phi(\tau) = \phi(0) + \phi'(0)(\tau + \frac{1}{2}\frac{\phi''(0)}{\phi'(0)}\tau^2 + ...) \quad (63)$$

Thus, plugging in $\int d\tau(\epsilon'')^2 \mapsto \int d\tau(\frac{\phi''}{\phi'})^2$. This, up to a total derivative, gives the action as

$$S = -N\frac{\alpha}{J}\int d\tau(\frac{\phi'''}{\phi'} - \frac{3}{2}(\frac{\phi''}{\phi'})^2) \quad (64)$$

With this action S, we finally get

$$\langle Z(\beta+it)\rangle_J = \quad (65)$$
$$\int_X d\Omega \exp\{\frac{N}{(\beta+it)J}\int Sch(\phi,\tau)d\tau\} \quad (66)$$

where $Sch(\phi,\tau)$ is the Schwartzian derivative $\frac{\phi'''}{\phi'} - \frac{3}{2}(\frac{\phi''}{\phi'})^2$ and $\phi(\tau) \in X = Diff(S^1)/SL(2,\mathbb{R})$.

Remark 3. *This functional integral can be computed exactly via the Duistermaat-Heckman theorem. It is an integral over a symplectic manifold of an exponential of a generator of a $U(1)$-symplectomorphism of that manifold. The integrals are then easy to compute:*

$$\langle Z(\beta+it)\rangle_J \quad (67)$$
$$= \int_X d\Omega \exp\{\frac{N}{(\beta+it)J}\int Sch(\phi,\tau)d\tau\} \quad (68)$$
$$= \frac{1}{[(\beta+it)J]^{\frac{3}{2}}}e^{\frac{N}{(\beta+it)J}} \quad (69)$$

This action is exactly the theory of our 2-dimensional black hole; the entire Dilaton gravity theory is exactly this integral over the Schwartzian derivative. This AdS/CFT calculation establishes the link between these fermions and the black hole.

The partition function $\langle Z(\beta+it)\rangle_J$ at large values of t behaves proportionally to $t^{-\frac{3}{2}}$. Squaring this expression yields t^{-3}, and it turns out that this explains the region of the curve before the dip time quite closely, so therefore this region can be explained gravitationally. It remains to explain the region of the curve post dip time. This can be explained by random matrix theory (RMT).

VII. ANALYTIC RAMP FOR A GUE RANDOM MATRIX HAMILTONIAN

If we compute $\langle Z\rangle$ for an $(L \times L)$ random matrix as our Hamiltonian, the curve to the right of the dip time looks quite similar to the same region in Figure 2. The SYK model has a particle-hole symmetry

$$P = K\prod_{i=1}^{N_d}(\bar{c}_i + c_i) \quad (70)$$

with K an anti-linear operator.

Remark 4. *The ensemble of the random matrix depends on N; it is periodic mod 8:*

1. *When $N \equiv$2 or 6 mod(8), P symmetry maps even and odd parity sectors to each other. The sectors do not have anti-lineary symmetry, and the corresponding ensemble is GUE.*
2. *When $N \equiv 0$ mod(8), P maps the sectors to themselves with $P^2 = 1$. The corresponding ensemble is GOE.*
3. *When $N \equiv 4$ mod(8), P still maps the sectors to themselves, but this time with $P^2 = -1$. The corresponding ensemble is GSE.*

Remark 5. *While to the left of the dip time the curve for a random matrix doesn't agree with that of the SYK result, it has the same behavior of t^{-3}. The reason for this is that the behavior can just be described by the mean density of eigenvalues, which vanishes like a square root for the random matrix. Plugging in this vanishing behavior to compute Z translates to the power $\beta^{-\frac{3}{2}}$, so the square root vanishing of the density of states gives similar behavior. Below we make this explicit.*

The GUE ensemble of Hermitian matrices M of rank L has the ensemble average given by

$$Z_{GUE} = \int\prod_{i,j}dM_{ij}\exp\{-\frac{L}{2}Tr[M^2]\} \quad (71)$$

Remark 6. *The natural perturbative parameter in SYK is $\frac{1}{N}$, while in RMT we expand in $\frac{1}{L} \sim e^{-N}$*

The partition function for such an M is then

$$Z(\beta,t) = Tr[e^{-\beta M - iMt}] \tag{72}$$

with the usual spectral form factor

$$g(t) = \frac{\langle Z(\beta+it)Z(\beta-it)\rangle_J}{\langle Z(\beta)\rangle_J^2} \tag{73}$$

In the large L limit, we can describe the eigenvalues by physical density $\rho(\lambda)$ and unit normalized density $\tilde{\rho}(\lambda) := \frac{\rho(\lambda)}{L}$. To make this precise, we have

$$\int d\lambda \rho(\lambda) = L \tag{74}$$

$$\int d\lambda \tilde{\rho}(\lambda) = 1 \tag{75}$$

This gives us a new way to write Z_{GUE}:

$$Z_{GUE} = \int \mathcal{D}\tilde{\rho}(\lambda) e^{-S}, \tag{76}$$

$$S = -\frac{L^2}{2}\int d\lambda \tilde{\rho}(\lambda)\lambda^2 \tag{77}$$

$$+L^2 \int d\lambda_1 d\lambda_2 \tilde{\rho}(\lambda_1)\tilde{\rho}(\lambda_2) \log|\lambda_1 - \lambda_2| \tag{78}$$

For large L, S has a saddle point given by the Wigner semicircle law:

$$\langle \rho(\lambda)\rangle_{GUE} = \tilde{\rho}_s(\lambda) \equiv \frac{1}{2\pi}\sqrt{4-\lambda^2} \tag{79}$$

Remark 7. *Notice that here the average eigenvalue spacing is* $\frac{1}{L}$.

In plotting the spectral form factor along time, we similarly get a slope, dip, ramp, and plateau. Before the dip time, $g(t)$ is dominated by the disconnected piece given by

$$g_d(t) = \frac{\langle Z(\beta+it)\rangle_J \cdot \langle Z(\beta-it)\rangle_J}{\langle Z(\beta)\rangle_J^2} \tag{80}$$

We first study the behavior when this term dominants (pre-dip time). To simplify calculations, we work in infinite temperature $\beta = 0$, although the resulting expression is also true at finite temperature. We have

$$\langle Z(0+it)\rangle_{GUE} = \int_{-2}^{2} d\lambda L \tilde{\rho}_s(\lambda) e^{-i\lambda t} = \frac{LJ_1(2t)}{t} \tag{81}$$

where J_1 is a Bessel function of the first kind. At late times the partition function decays as $\frac{L}{t^{3/2}}$, so we get

$$g_d(t) = \frac{|\langle Z(0+it)\rangle_J|^2}{L^2} \tag{82}$$

$$\sim \frac{1}{t^3} \tag{83}$$

Thus the late time decay of $g(t)$ pre-dip time is $\sim \frac{1}{t^3}$. This is because $\tilde{\rho}_s(\lambda)$ vanishes as a square root near the edge of the spectrum.

Post-dip time, $g(t)$ is dominated by

$$g_c(t) := g(t) - g_d(t) \tag{84}$$

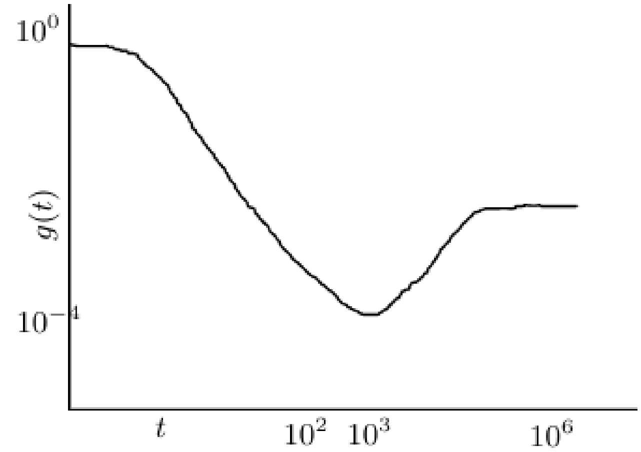

FIG. 3: log-log plot of $g(t)$ for GUE random matrices. Produced in[1] by treating two samples of the eigenvalues of $2^{12}-$dimensional matrices as a single sample of a $2^{13}-$dimensional matrix. Source: A. Lau.

Remark 8. *In this post-dip time domain, $g(t)$ and $g_c(t)$ are almost equal, but $g_c(t)$'s ramp extends to very early times, giving better perturbative control.*

Again working in infinite temperature ($\beta = 0$), we write

$$g_c(t) = \int d\lambda_1 d\lambda_2 R_2(\lambda_1,\lambda_2) e^{i(\lambda_1-\lambda_2)t} \tag{85}$$

where we define

$$R_2(\lambda_1,\lambda_2) := \langle \delta\tilde{\rho}(\lambda_1)\delta\tilde{\rho}(\lambda_2)\rangle_{GUE} \tag{86}$$

is the connected pair correlation function of the unit-normalized density $\tilde{\rho}$, with $\delta\tilde{\rho}(\lambda) = \tilde{\rho}(\lambda) - \tilde{\rho}_s(\lambda)$ the fluctuation around the mean eigenvalue via the Wigner semicircle law.

Definition 6. *It is a result of RMT that near the center of the Wigner semicircle $R_2(\lambda_1,\lambda_2)$ is given by the square of the **sine kernel** plus a delta function:*

$$R_2(\lambda_1,\lambda_2) = -\frac{\sin^2(L[\lambda_1-\lambda_2])}{(\pi L[\lambda_1-\lambda_2])^2} + \frac{1}{L\pi}\delta(\lambda_1-\lambda_2) \tag{87}$$

Notice that our expression for $g_c(t)$ is a Fourier transform. Taking the Fourier transform of both of these terms, we get

$$g_c(t) \sim \frac{t}{2\pi L^2} \text{ for } t < 2L, \tag{88}$$

$$\sim \frac{1}{\pi L} \text{ for } t \geq 2L \tag{89}$$

This explains the plot in Figure 3.

Remark 9. *One can do this without an appeal to the sine kernel; one can derive*

$$R_2(\lambda_1,\lambda_2) \approx -\frac{1}{2(\pi L(\lambda_1-\lambda_2))^2} \tag{90}$$

perturbatively from the action.

VIII. ANALYTIC RAMP FOR SYK

To analytically derive the ramp behavior of the SYK model, we start with the general definition of ZZ^*:

$$\langle Z(\beta+it)Z(\beta-it)\rangle \tag{91}$$

$$= \int d\lambda_1 d\lambda_2 \langle\rho(\lambda_1)\rho(\lambda_2)\rangle e^{-\beta(\lambda_1+\lambda_2)}e^{-i(\lambda_1-\lambda_2)t} \tag{92}$$

We take GUE statistics

$$\langle\rho(\lambda_1)\rho(\lambda_2)\rangle = \langle\rho(E)\rangle\delta(\lambda_1-\lambda_2) \tag{93}$$

$$+\langle\rho(\lambda_1)\rangle\langle\rho(\lambda_2)\rangle(1-\frac{\sin^2[\pi\langle\rho(E)\rangle(\lambda_1-\lambda_2)]}{[\pi\langle\rho(E)\rangle(\lambda_1-\lambda_2)]^2}) \tag{94}$$

This gives us

$$\langle Z(\beta+it)Z(\beta-it)\rangle = |\langle Z(\beta+it)\rangle|^2 \tag{95}$$

$$+\int dEe^{-2\beta E}min(\frac{t}{2\pi},\langle\rho(E)\rangle) \tag{96}$$

Next we use the effective Schwartzian derivative theory:

$$Z(\beta)=\int_X d\Omega\exp\{-\frac{N\pi\alpha_S}{\beta\mathcal{J}}\int_0^{2\pi}Sch(\phi,\tau)d\tau\} \tag{97}$$

where α_S is a numerical coefficient depending on $q=4$, X is the symplectic manifold $Diff(S^1)/SL(2,\mathbb{R})$, and $\mathcal{J}$ sets the scale of the Hamiltonian for different values of q.

Definition 7. *We define the* ***specific heat*** *c by*

$$c=\frac{4\pi^2\alpha_S}{\mathcal{J}} \tag{98}$$

The classical and 1-loop contributions to this action are given by

$$Z^{1-loop}(\beta)=\frac{\#}{(\beta\mathcal{J})^{3/2}}\exp\{\frac{2\pi^2N\alpha_S}{\beta\mathcal{J}}\} \tag{99}$$

Remark 10. *The theory turns out to be 1-loop exact. The proof of this is found in*[4].

We take into account the soft mode integral by dividing the saddle-point expression for the partition function by a factor of $(\beta+it)^{3/2}$ (in the large N limit). The large N free energy limit in the holographic is given by

$$\log Z = N(\epsilon_0\beta+s_0+\frac{c}{2\beta}) \tag{100}$$

we find that the disconnected term $g_d(t)$ contributes

$$\frac{|\langle Z(\beta+it)\rangle|^2}{\langle Z(\beta)\rangle^2}=\frac{\beta^3}{(\beta^2+t^2)^{3/2}}\exp\{-\frac{cNt^2}{\beta(\beta^2+t^2)}\} \tag{101}$$

Observe that the dependence on the exponent becomes negligible when at $t\approx>\sqrt{N}$ and we have a power law decay $\sim\frac{1}{t^3}$.

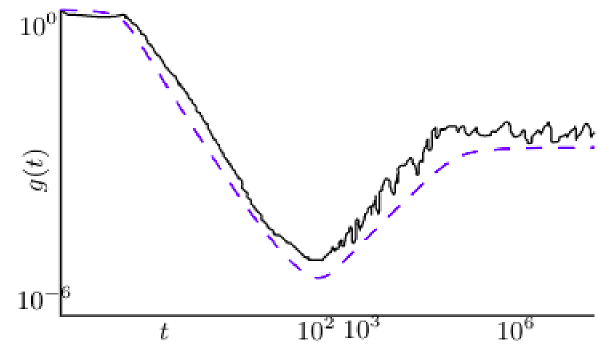

FIG. 4: log-log plot of $g(t)$ for the exact diagonalization solution and the numerical solution. Source: A. Lau.

Remark 11. *While this behaves similarly to the GUE random matrix Hamiltonian above, the actual numerical solution does not match that of the SYK model. However, the ramp matches quite similarly, as we express below.*

To evaluate the second term in (70), we use the solution $S(E)$ of the Schwinger-Dyson equations. In the holographic limit, however, we have

$$S(E)=Ns_0+\sqrt{2c(E-E_0)N} \tag{102}$$

When we neglect 1-loop factors from the integral over E, we get

$$g_{ramp}(t)\sim\frac{t}{2\pi}\exp[-2Ns_0-\frac{cN}{\beta}] \tag{103}$$

for $\frac{t}{2\pi}<e^{Ns_0}$,

$$g_{ramp}(t)\sim\frac{t}{2\pi}\exp[-2Ns_0-\frac{cN}{\beta}-\frac{\beta}{cN}\log^2(\frac{t}{2\pi e^{Ns_0}})] \tag{104}$$

for $e^{Ns_0}<\frac{t}{2\pi}<\frac{t_p}{2\pi}$, and

$$g_{ramp}(t)\sim\exp[-Ns_0-\frac{3cN}{4\beta}] \tag{105}$$

for $t_p<t$, where $t_p=2\pi e^{Ns_0+\frac{cN}{2\beta}}$. This numerical solution looks fairly close to the exact solution mentioned above, but it is unknown how the true large N answer for $g(t)$ behaves. In Figure 4 the numerical and exact solutions are sketched.

IX. CONCLUSION

We conclude that the late time behavior of the SYK model, and therefore late time behavior of horizon fluctuations in large anti-de Sitter (AdS) black holes, are governed by random matrix dynamics, a hallmark of a quantum chaotic system.

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[2] Brustein, R., Zigdon Y. "Revealing the interior of black holes out of equilibrium in the Sachdev-Ye-Kitaev model." *Phys. Rev. D* **98**, 066013. Sep. 24, 2018.

[3] Maldacena, J., Stanford, D. "Remarks on the Sachdev-Ye-Kitaev model," *Phys. Rev. D* **94**, 106002. Nov. 4, 2016..

[4] Stanford, D., Witten, E. "Fermionic localization of the schwarzian theory." *J. High Energ. Phys.* **2017**, 8 (2017). https://doi.org/10.1007/JHEP10(2017)008