Topological Quantum Computation Problems

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In spring 2018, Shawn X. Cui taught an excellent seminar on topological quantum computation, and these were the homework problems.

Contents

1 Logical operators in toric code

Question. *In class, we studied string operators $S^Z(t)$ and $S^Z(t')$ where t and t' are string operators on the lattice and dual lattice, respectively. By definition, $S^Z(t)$ acts by Pauli Z on each edge of t and by identity otherwise. Similarly, $S^X(t')$ acts by Pauli X on each edge crossed by t' and by identity otherwise. Consider the case where both t, t' are closed strings. Let V_{gs} be the ground state space.*

- *Show that $S^Z(t)$ and $S^X(t')$ preserve V_{gs} for arbitrary closed strings t, t'. Moreover, show that the action of these operators on V_{gs} only depends on the isotopy class of the strings. In particular, this means if a closed string is contractible, the corresponding string operator acts by identity on ground states.*

- *By the previous result, there are four string operators of $Z-$type which are $\{S^Z(\emptyset), S^Z(m), S^Z(l), S^Z(m \cup l)\}$, where $\emptyset$ is the empty string or any contractible string, m is a loop along the horizontal direction, and l is a loop along the vertical direction. See Figure 4. Similarly, there are four strings of $X-$type, $\{S^X(\emptyset), S^X(m), S^X(l), S^X(m \cup l)\}$. Let*

$$\hat{Z}_1 = S^Z(m), \hat{Z}_2 = S^Z(l), \tag{1}$$
$$\hat{X}_1 = S^X(l'), \hat{X}_2 = S^X(m') \tag{2}$$

Show that on the ground states the commutation relations between the operators $\{\hat{Z}_1, \hat{Z}_2, \hat{X}_1, \hat{X}_2\}$ behave like the usual Pauli operators $\{Z_1, Z_2, X_1, X_2\}$. These operators are the logical operators.

- *Show that the space of logical operators, i.e. those preserving V_{gs}, is generated as an algebra by $\{\hat{Z}_1, \hat{Z}_2, \hat{X}_1, \hat{X}_2\}$. (Hint: the space of all operators on a physical qubit has a basis given by $\{Id, X, Z, XZ\}$.)*

Proof.

- The Hamiltonian on toric code is given by

$$H := \sum_{v \in V} (1 - A_v) + \sum_{p \in F} (1 - B_p) \tag{3}$$

for

$$A_v := (\otimes_{e \in star(v)} X) \otimes (\otimes_{e \in E - star(v)} Id), \tag{4}$$
$$B_p := (\otimes_{e \in \partial p} Z) \otimes (\otimes_{e \in E - \partial p} Id) \tag{5}$$

and F the set of plaquettes, E is the set of edges, and V the set of vertices. Thus the ground state V_{gs} is given by

$$V_{gs} = \{|\psi\rangle \in \mathscr{H}_{T^2} = \otimes_{e \in E} \mathbb{C}^2 : A_v |\psi\rangle = |\psi\rangle, B_p |\psi\rangle = |\psi\rangle, \forall v \in V, p \in F\} \tag{6}$$

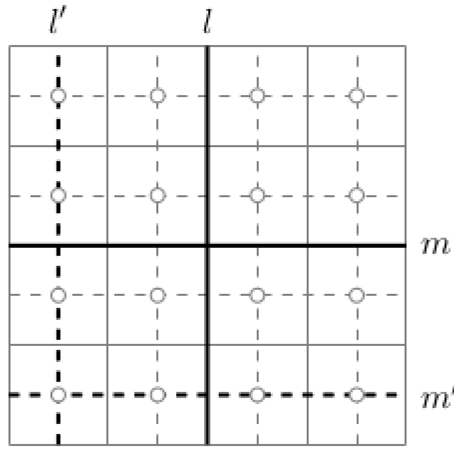

Figure 1: Closed strings in the lattice and dual lattice on the torus.

First off, we examine the commutators $[A_v, S^Z(t)]$, $[B_p, S^Z(t)]$, $[A_v, S^X(t')]$, and $[B_p, S^X(t')]$ for t, t' closed loops. If t is a closed loop, every vertex in t must be connected to an even number of edges in t; a vertex in t connected to an odd number of edges in t would be a boundary of t, which is supposed to be closed. If v is a vertex that isn't in t, then A_v must commute with $S^Z(t)$, as they are acting on different tensor factors. If v is a vertex in t, it is adjacent to either 2 or 4 edges in t. Thus every vertex in t has an even number of Z operators in the tensor product. By inspection, $XZ = -ZX$, so we have $A_v S^Z(t) = (-1)^{2,4} S^Z(t) A_v = S^Z(t) A_v$, i.e. $[A_v, S^Z(t)] = 0, \forall v \in t$ as well. Furthermore, since B_p only consists of Z operators and identity operators, and so does $S^Z(t)$, $[B_p, S^Z(t)]$ must be 0 for all $p \in F$.

Similarly, for a plaquette $p \in F$ in t', there can either be 2 or 4 dual edges in p, and thus either 2 or 4 *edges* in ∂p. By the same reasoning as above, $S^X(t') B_p = (-1)^{2,4} B_p S^X(t') = B_p S^X(t')$, so $[B_p, S^X(t')] = 0, \forall p \in t'$. Similarly, A_v is comprised only of X operators and identity operators, and so is $S^X(t')$, so $[A_v, S^X(t')]$ must be 0 for all $p \in F$. Note that this is true independently of the closed strings t, t'.

Let $|\psi\rangle$ be a ground state, and define $|\phi\rangle := S^Z(t)\,|\psi\rangle$, $|\phi'\rangle := S^X(t')\,|\psi\rangle$.

From before, we have

$$A_v |\phi\rangle = A_v S^Z(t) |\psi\rangle = S^Z(t) A_v |\psi\rangle = S^Z(t) |\psi\rangle = |\phi\rangle \tag{7}$$

$$B_p |\phi\rangle = B_p S^Z(t) |\psi\rangle = S^Z(t) B_p |\psi\rangle = S^Z(t) |\psi\rangle = |\phi\rangle \tag{8}$$

and

$$A_v |\phi'\rangle = A_v S^X(t') |\psi\rangle = S^X(t') A_v |\psi'\rangle = S^X(t') |\psi\rangle = |\phi'\rangle \tag{9}$$

$$B_p |\phi'\rangle = B_p S^X(t') |\psi\rangle = S^X(t') B_p |\psi\rangle = S^X(t') |\psi\rangle = |\phi'\rangle \tag{10}$$

Thus $S^Z(t), S^X(t')$ preserve V_{gs}, if t, t' are closed strings.

Consider $S^Z(t) |\psi\rangle$. We can deform the action of $S^Z(t)$ by acting by B_p on $S^Z(t)$ where at least one edge in ∂p is in t. This deforms t around the plaquette p, because it acts by Z on the edges around p where t wasn't, and cancels out the edges around p where t already was, because $Z^2 = Id$. Similarly, we can deform the path of t' by acting by A_v on $S^X(t')$ where at least one edge adjacent to v is crossed by an edge in t'. This deforms t' around the vertex v, because it acts by X on the dual edges around v where t' wasn't, and cancels out the dual edges around v where t' already was, by acting on such edges twice with X, and thus acting on such edges by the identity. See the Figure 2 for an example.

Thus if we get t_2 by a deformation on t_1, we have $S^Z(t_2) = B_{p_1} ... B_{p_n} S^Z(t_2)$ for some plaquettes $p_i, i \in \{1, ..., n\}$. Thus, for $|\psi\rangle$ a ground state, we have

$$S^Z(t_2) |\psi\rangle = B_{p_1} ... B_{p_n} S^Z(t_1) |\psi\rangle = S^Z(t_1) |\psi\rangle \tag{11}$$

$$S^X(t_2') |\psi\rangle = A_{v_1} ... A_{v_n} S^X(t_1') |\psi\rangle = S^X(t_1') |\psi\rangle \tag{12}$$

so although the operators S^X, S^Z change with isotopy, their action on V_{gs} is preserved.

Up to isotopy, m intersects l' on only one edge of the lattice, as well as l and m'. Thus the commutation relations between $\hat{Z}_1, \hat{X}_1$ and $\hat{Z}_2, \hat{X}_2$ come down to their action on that one edge ($\hat{Z}_1$ and $\hat{X}_2$ need not intersect, and the same goes for $\hat{Z}_2$ and $\hat{X}_1$). Since their actions are Z and X, they must obey the same commutation relations as $\{Z_1, Z_2, X_1, X_2\}$.

Since the space of all operators on a qubit is generated by $\{Z, X\}$, and $\{\hat{Z}_1, \hat{Z}_2, \hat{X}_1, \hat{X}_2\}$ is isomorphic as an algebra to $\{Z_1, Z_2, X_1, X_2\}$, the space of logical operators is generated by $\{\hat{Z}_1, \hat{Z}_2, \hat{X}_1, \hat{X}_2\}$. $\square$

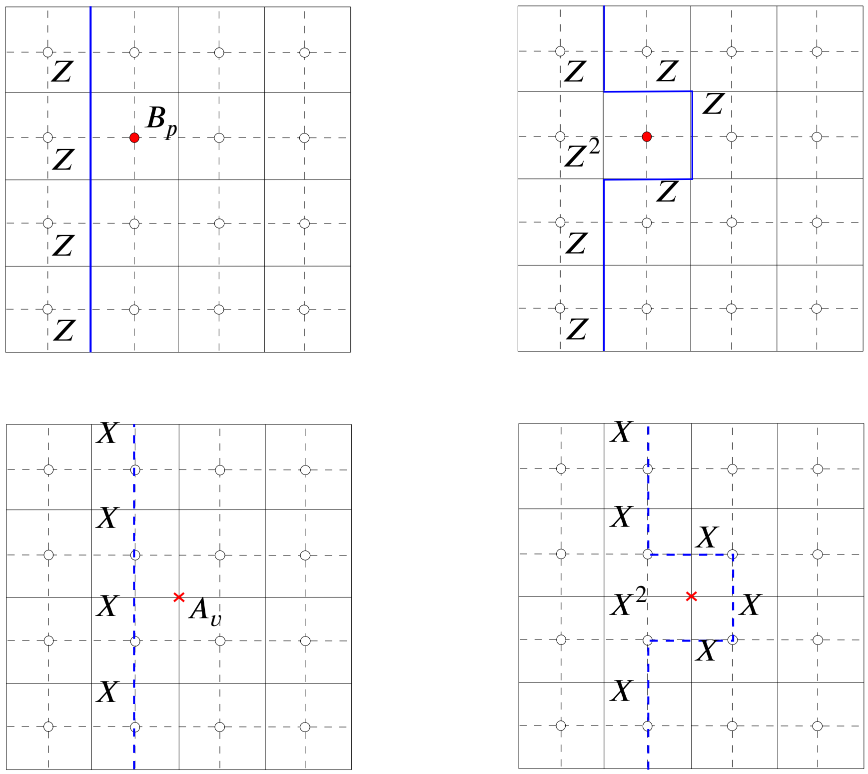

Figure 2: Deformation of loops.

2 V_{gs} is an error-correcting code

Question. *Let the square lattice $\mathscr{L}$ in the definition of toric code have size $L \times L$, namely, there are L edges in the shortest non-contractible loop both along the horizontal direction and along the vertical direction. Let*

$$P := \prod_{v \in V} \frac{Id + A_v}{2} \prod_{p \in F} \frac{Id + B_p}{2} \tag{13}$$

Namely, P is the projector onto the ground space V_{gs}. Let $\mathscr{O}$ be any operator acting on less than L qubits, namely, $\mathscr{O}$ acts nontrivially on at most $L - 1$ qubits. Show *that*

$$P\mathcal{O}P = \alpha_{\mathcal{O}}P, \tag{14}$$

for some scalar $\alpha_{\mathcal{O}}$*. (*V_{gs} *is an error-correcting code which corrects errors on arbitrary* $\lfloor\frac{L-1}{2}\rfloor$ *qubits. (Hint: it suffices to show this equation for a basis of the space of operators acting on at most* $L-1$ *qubits. A basis for this space is given by*

$$\{\prod_{e\in E}\mathscr{P}_e : \mathscr{P}_e \in \{Id, X, Z, XZ\},\ \textit{and at most } L-1\ \mathscr{P}_e'\textit{s are not trivial}\} \tag{15}$$

Proof. Each edge in $\mathscr{L}$ is the side of two plaquettes and belongs to the star of two vertices. Thus, for each edge qubit e in some state $[... \otimes e \otimes ...]$, we have

$$(\frac{2Id}{2})^{n_6}(\frac{Id+X}{2})(\frac{2Id}{2})^{n_5}(\frac{Id+X}{2})(\frac{2Id}{2})^{n_4}\cdot \tag{16}$$

$$(\frac{2Id}{2})^{n_3}(\frac{Id+Z}{2})(\frac{2Id}{2})^{n_2}(\frac{Id+Z}{2})(\frac{2Id}{2})^{n_1} \tag{17}$$

acting on e, with $n_i \in \{0, ..., L^2-2\}$ depending on the order of ennumerating the vertices and plaquettes. This action on each e becomes

$$(\frac{Id+X}{2})(\frac{Id+X}{2})(\frac{Id+Z}{2})(\frac{Id+Z}{2}) = (\frac{Id+X}{2})(\frac{Id+Z}{2}) \tag{18}$$

$$= (\frac{Id+X+Z+XZ}{4}) := P_e \tag{19}$$

For each edge, we have

$$P_e Id P_e = \frac{Id+X+Z+XZ}{8} = \frac{1}{2}P_e \tag{20}$$

$$P_e X P_e = P_e P_e = \frac{Id+X+Z+XZ}{8} = \frac{1}{2}P_e \tag{21}$$

$$P_e Z P_e = \frac{Id+X+Z+XZ}{8} = \frac{1}{2}P_e \tag{22}$$

$$P_e XZ P_e = -\frac{Id+X+Z+XZ}{8} = -\frac{1}{2}P_e \tag{23}$$

Thus, tensoring all the P_es together to form P, we get

$$P\mathcal{O}P = \alpha_{\mathcal{O}}P \tag{24}$$

where $\alpha_{\mathcal{O}}$ is a product of scalar multiples of $\frac{1}{2}$. □

3 Braiding statistics of quasi-particles in toric code

Question. *In class, we have shown that there are four types of quasi-particles, the vacuum 1, the electric charge e, the magnetic charge m, and the composite em of an electric charge with a magnetic charge. Consider a pair of electric charges e, and denote the state of such configuration by*

$$|\psi_{in}\rangle = S^Z(t)\,|\epsilon\rangle \tag{25}$$

where $|\epsilon\rangle$ is some ground state. If we swap the two particles in counterclockwise direction, then the state becomes

$$|\psi_{fi}\rangle = S^Z(t')\,|\epsilon\rangle \tag{26}$$

But since t and t' can be deformed to each other, we have $|\psi_{in}\rangle = |\psi_{fi}\rangle$. Hence the electric charge e is a boson. Similarly, the magnetic charge m is also a boson. However, show that the composite em is a fermion.

Proof. I assume that an *em* charge on the lattice is a site (adjacent vertex and dual vertex) on the lattice. Exchanging the *em* sites involves composing the paths creating the quasiparticles with a loop where the vertices of said loop are the locations of the particles:

$$\begin{aligned}
|\psi_{em}\rangle := S^X(t')S^Z(t)\,|\epsilon\rangle \xrightarrow{exchange} & S^X(t' \cup t'_{loop})S^Z(t \cup t_{loop})\,|\epsilon\rangle && (27)\\
=& S^X(t')S^X(t'_{loop})S^Z(t)S^Z(t_{loop})\,|\epsilon\rangle && (28)\\
=& S^X(t')S^X(t'_{loop})S^Z(t)\,|\epsilon\rangle && (29)\\
=& -S^X(t')S^Z(t)S^X(t'_{loop})\,|\epsilon\rangle && (30)\\
=& -S^X(t')S^Z(t)\,|\epsilon\rangle && (31)\\
=& -|\psi_{em}\rangle && (32)
\end{aligned}$$

since trivial (dual) loops act by identity on $|\epsilon\rangle \in V_{gs}$, and $S^Z(t)$ intersects $S^X(t'_{loop})$ once, since the vertices are adjacent, and thus they anticommute. Since the exchange operator gives a phase factor of -1, *em* charges are fermions. □

4 Single-particle excitation on a torus

Recall Kitaev's quantum double model based on a finite group G on a torus. For $\mathscr{L}$ an arbitrary lattice on the torus, we fix an orientation and associated to each edge

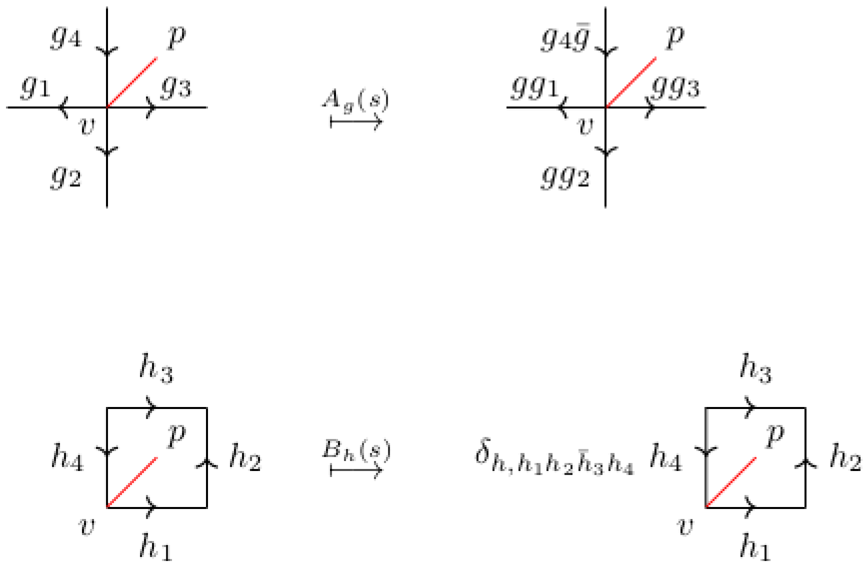

Figure 3: Operators used to construct the Hamiltonian, for $g, h \in G$

the Hilbert space $\mathbb{C}[G]$ for a total Hilbert space on $\mathscr{L}$ denoted by $\mathscr{H}_{tot}$. We denote the set of all vertices V and the set of all plaquettes F. For each site $s = (v, p)$ (for each vertex we associate the plaquette to its upper right) we have the operators: We further define

$$A(v) := \frac{1}{|G|} \sum_{g \in G} A_g(v), B(p) = B_e(v, p) \tag{33}$$

and then define the Hamiltonian by

$$H = \sum_{v \in V} (1 - A(v)) + \sum_{p \in F} (1 - B(p)) \tag{34}$$

where the ground state is

$$V_{gs} = \{ |\psi\rangle \in \mathscr{H}_{tot} \,:\, A(v)\,|\psi\rangle = |\psi\rangle \,, B(p)\,|\psi\rangle = |\psi\rangle \} \tag{35}$$

Question. *Single-particle excitations cannot occur on the sphere, but they can occur on the torus. Consider a square lattice on the torus. All horizontal edges are oriented to the right and all vertical edges are oriented upward. Let G be* *a finite group and let $a, b \in G$ be two group elements which do not commute. Let $r = aba^{-1}b^{-1}$. Recall that on each edge lives a Hilbert space with the basis $\{|g\rangle : g \in G\}$ and the total Hilbert space is the tensor product of the Hilbert space on all edges. Let $|\psi\rangle$ be the basis state in the total Hilbert space whose value at each edge is shown in Figure 7, and all other edges are labeled by e. Define*

$$|\psi_{a,b}\rangle := \prod_{v \in V} A(v)\,|\psi\rangle \tag{36}$$

1. *By definition, $|\psi_{a,b}\rangle$ is stabilized by all $A(v)$s. Let p_0 be the plaquette on the top right of the lattice. Show that*

$$B(p)\,|\psi_{a,b}\rangle = |\psi_{a,b}\rangle\,, \forall p \neq p_0, \tag{37}$$

$$B(p_0)\,|\psi_{a,b}\rangle = 0 \tag{38}$$

   Thus $|\psi_{a,b}\rangle$ is a state which violates only one constraint. Note that $|\psi_{a,b}\rangle$ is not the zero vector.

2. *Let C be the conjugacy class containing r. Let v_0 be a vertex on the boundary of p_0 and $s_0 = (v_0, p_0)$ be a site. For each $c \in C$, define*

$$|c\rangle := B_c(s_0)\,|\psi_{a,b}\rangle \tag{39}$$

   and let $V = span\{|c\rangle : c \in C\}$. Show that the states $\{|c\rangle : c \in C\}$ form a basis of V.

3. *It is not hard to see that any state in V is stabilized by all $A(v)$ and $B(p)$ for which $v \neq v_0, p \neq p_0$. What is the action of the operators $A_g(s_0)$ and $B_h(s_0)$ on V? Write it out under the basis $\{|c\rangle : c \in C\}$. Conclude which irrep V corresponds to. A state in V represents an excitation on the single site s_0.*

Proof. 1. Every edge in $\mathscr{L}$ is hit twice by $\prod_{v \in V}$. Due to the orientation of each edge around every plaquette, when we order the edges clockwise around the plaquette we get right multiplication by $\overline{g}$ on one edge, and left multiplication by g on the right edge, for every g in the sum in $A(v)$, once all vs are taken into account. Suppose a plaquette p's state $|p\rangle$ has edges h_1, h_2, h_3, and h_4 going clockwise around the plaquette, starting from the bottom edge.

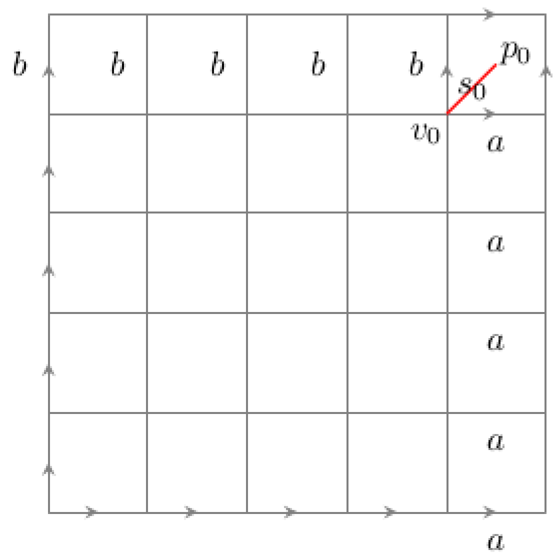

Figure 4: Lattice on a torus

Acting on $\mathscr{L}$ by $\prod_{v\in V} A(v)$, the Hilbert subspace around the plaquette becomes

$$\prod_{v\in V} A(v)\,|\psi\rangle = \frac{1}{|G|^4}\sum_{g_1,g_2,g_3,g_4}(g_1h_1\overline{g_2}\otimes g_2h_2\overline{g_3}\otimes g_4h_3\overline{g_3}\otimes g_1h_4\overline{g_4}) \tag{40}$$

This gives us

$$B(p)\,|\psi_{a,b}\rangle := B_e(p)\,|\psi_{a,b}\rangle = \frac{1}{|G|^4}\sum_{g_1,g_2,g_3,g_4}\delta_{e,g_1h_1\overline{g_2}g_2h_2\overline{g_3}g_3\overline{h_3}\,\overline{g_4}g_4\overline{h_4}\,\overline{g_1}}\,|\psi_{a,b}\rangle \tag{41}$$

$$= \frac{1}{|G|^4}|G|^3\sum_{g_1\in G}\delta_{e,g_1h_1h_2\overline{h_3}\,\overline{h_4}\,\overline{g_1}}\,|\psi_{a,b}\rangle \tag{42}$$

In our particular labelling, all plaquettes except for p_0 are of configuration either *eeee*, *aeae*, or *ebeb*, so the action of $B(p)$ for all $p \in F$ except for p_0 is the identity.

The configuration on p_0 is $abab$, giving us

$$B(p_0)\,|\psi_{a,b}\rangle = \frac{1}{|G|}\sum_{g\in G}\delta_{e,gab\overline{a}\,\overline{b}\,\overline{g}}\,|\psi_{a,b}\rangle \tag{43}$$

Since a, b do not commute, $e \neq gab\overline{a}\,\overline{b}\,\overline{g}$ for any $g \in G$, and the state becomes 0.

2. From the above calculation, we have

$$|c\rangle := B_c(s_0)\,|\psi_{a,b}\rangle = \frac{1}{|G|}\sum_{g\in G}\delta_{c,gr\overline{g}}\,|\psi_{a,b}\rangle \tag{44}$$

There is a unique set of $g \in G$ such that, for a fixed $c \in C$, $gr\overline{g} = c$. Call this set G_c. Thus completely disjoint subsets of G are kept in the sum for each $c \in C$.
Let $\{a_c \in \mathbb{C} | c \in C\}$ be such that

$$\sum_{c\in C} a_c\,|c\rangle = 0 = \sum_{c\in C} a_c \prod_{v\in V}\frac{1}{|G|}\sum_{g\in G_c} A_g(v)\,|\psi\rangle \tag{45}$$

But since these are all different gs, the only $\{a_c\}$ set in which this is true is $a_c = 0$ for all $c \in C$.

3. Fix a $c \in C$ for now. We have

$$\begin{aligned}
A_g(s_0)\,|c\rangle &= A_g(s_0)B_c(s_0)\,|\psi_{a,b}\rangle && (46)\\
&= \delta_{gc\overline{g},gab\overline{a}\,\overline{b}\,\overline{g}}A_g(s_0)\,|\psi_{a,b}\rangle && (47)\\
&= B_{gc\overline{g}}A_g(s_0)\,|\psi_{a,b}\rangle && (48)\\
&= B_{gc\overline{g}}A_g(s_0)\prod_{v\in V}\frac{1}{|G|}\sum_{g'\in G}A_{g'}\,|\psi\rangle && (49)\\
&= B_{gc\overline{g}}\,|\psi_{a,b}\rangle && (50)
\end{aligned}$$

since the action of $A_g(s_0)$ just rearranges the sum on $v_0 \in s_0$ for $\frac{1}{|G|}\sum_{g'\in G} A_{g'}\,|\psi\rangle$. Thus

$$A_g(s_0)\,|c\rangle \mapsto |gc\overline{g}\rangle \tag{51}$$

for all $c \in C$.

Next we look at $B_h(s_0)\,|c\rangle$. We have

$$B_h(s_0)\,|c\rangle = B_h(s_0)B_c(s_0)\,|\psi_{a,b}\rangle \tag{52}$$

For a plaquette p with clockwise labels g_1, g_2, g_3, and g_4, starting from the bottom label, we have

$$\begin{aligned} B_h(p)B_c(p) &= B_h(p)\delta_{c,g_1 g_2 \overline{g_3}\,\overline{g_4}} &(53)\\ &= \delta_{h,g_1 g_2 \overline{g_3}\,\overline{g_4}}\delta_{c,g_1 g_2 \overline{g_3}\,\overline{g_4}} &(54)\\ &= \delta_{h,c}\delta_{c,g_1 g_2 \overline{g_3}\,\overline{g_4}} &(55)\\ &= \delta_{h,c} B_c(p) &(56)\end{aligned}$$

Thus we have

$$B_h(s_0)\,|c\rangle = \delta_{h,c}\,|c\rangle \tag{57}$$

for all $c \in C$.

We now check what irreducible representation of the quantum double V corresponds to. An irreducible representation of the quantum double corresponds to (C, χ), where χ is an irreducible representation of the centralizer of r. The Hilbert space corresponding to (C, χ) is given by

$$\mathbb{C}[C] \otimes V_\chi \tag{58}$$

Since $V = \mathbb{C}[C]$, the irreducible representation corresponding to V is $(C, \mathbf{1})$. □

5 Local operators interpreted as ribbon operators

Question. *Let $s = (v, p)$ be any site on a lattice. We show the local operators $A_g(s)$ and $B_h(s), h, g \in G$ can be interpreted as ribbon operators for certain ribbons. We start with $B_h(s)$. Let t_s be a ribbon contained in the plaquette p, starting and ending both at s. See Figure 5 (Left). It consists of four triangles of type-II (direct triangles) t_1, t_2, t_3, t_4, and is directed in the order the triangles are listed. Assume the edges on the boundary of p are directed as shown in Figure 5 (Left) and a basis state $|x_1, x_2, x_3, x_4\rangle$ is given. Then*

$$F^{(h,g)}(t_i)\,|x_i\rangle = \delta_{g,x_i}\,|x_i\rangle \tag{59}$$

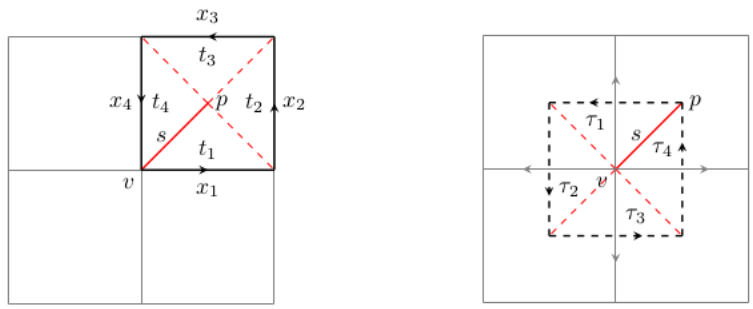

Figure 5: Lattice on a torus

By the inductive formula for ribbon operators

$$F^{(h,g)}(t_1t_2) := \sum_{k\in G} F^{(h,k)}(t_1)F^{(\bar{k}hk,\bar{k}g)}(t_2), \tag{60}$$

we have

$$F^{(h,g)}(t_1t_2)\,|x_1,x_2\rangle = \sum_{k\in G} F^{(h,k)}(t_1)\,|x_1\rangle \otimes F^{(\bar{k}hk,\bar{k}g)}(t_2)\,|x_2\rangle \tag{61}$$

$$= \sum_{k\in G} \delta_{k,x_1}\delta_{\bar{k}g,x_2}\,|x_1,x_2\rangle \tag{62}$$

$$= \delta_{g,x_1x_2}\,|x_1,x_2\rangle \tag{63}$$

Inductively, it is not hard to see that

$$F^{(h,g)}(t_s)\,|x_1,x_2,x_3,x_4\rangle = \delta_{g,x_1x_2x_3x_4}\,|x_1,x_2,x_3,x_4\rangle = B_g(s) \tag{64}$$

Similarly, let τ_s be a ribbon around the vertex v, starting and ending at s. It has four triangles of type-I (dual triangles) $\tau_1,\tau_2,\tau_3,\tau_4$, and is also directed in the order the triangles are listed. See Figure 5 (Right). Prove that

$$F^{(h,g)}(\tau_s) = \delta_{g,e}A_h(s). \tag{65}$$

Note that $A_h(s)$ actually only depends on v, hence the ribbon operator $F^{(h,g)}(\tau_s)$ does not depend on the choice of the initial site.

Proof. By the inductive formula for ribbon operators, we have

$$F^{(h,g)}(\tau_s) = \sum_{k\in G} F^{(h,k)}(\tau_1\tau_2\tau_3)F^{(\bar{k}hk,\bar{k}g)}(\tau_4) \tag{66}$$

$$= \sum_{k\in G}\sum_{l\in G} F^{(h,l)}(\tau_1\tau_2)F^{(\bar{l}hl,\bar{l}k)}(\tau_3)F^{(\bar{k}hk,\bar{k}g)}(\tau_4) \tag{67}$$

$$= \sum_{k\in G}\sum_{l\in G}\sum_{m\in G} F^{(h,m)}(\tau_1)F^{(\overline{m}hm,\overline{m}l)}(\tau_2)F^{(\bar{l}hl,\bar{l}k)}(\tau_3)F^{(\bar{k}hk,\bar{k}g)}(\tau_4) \tag{68}$$

Since

$$F^{(h,g)}(t)\,|x\rangle = \delta_{g,e}\,|hx\rangle \tag{69}$$

This gives us

$$F^{(h,g)}(\tau_s)\,|x_1,x_2,x_3,x_4\rangle = \sum_{k\in G}\sum_{l\in G}\sum_{m\in G} \delta_{m,e}\,|hx_1\rangle \otimes \delta_{\overline{m}l,e}\,|\overline{m}hmx_2\rangle \tag{70}$$

$$\otimes\, \delta_{\bar{l}k,e}\,|\bar{l}hlx_3\rangle \otimes \delta_{\bar{k}g,e}\,|\bar{k}hkx_4\rangle \tag{71}$$

$$= |hx_1\rangle \otimes |hx_2\rangle \otimes |hx_3\rangle \otimes \delta_{g,e}\,|hx_4\rangle \tag{72}$$

$$= \delta_{g,e}A_h(s)\,|x_1,x_2,x_3,x_4\rangle \tag{73}$$

□

6 Excitation types can be locally measured

Question. *We know that an excitation in general occupies a site $s = (v,p)$ and the types of excitations are in one-to-one correspondence with irreps of DG, the quantum double of group G. Recall that the irreps $Irr(DG)$ are characterized by the pairs (C,χ), where C is a conjugacy class with a pre-selected element $r \in C$ and χ is an irrep of $Z(r)$, the centralizer of r. For each $c \in C$, arbitrarily choose $q_c \in G$ such that $q_c r\overline{q_c} = c$. Also recall that DG acts on the total Hilbert space by the local operators $D(s)$ (recall that $D(s)$ is the algebra generated by $A_g(s), B_h(s), g,h \in G$). We wish to find a set of elements*

$$\{P_{(C,\chi)} \in DG : (C,\chi) \in Irr(DG)\} \tag{74}$$

which satisfy the following properties.

$$P_{(C,\chi)}P_{(C',\chi')} = \delta_{C,C'}\delta_{\chi,\chi'}, \tag{75}$$

$$\sum_{(C,\chi)\in Irr(DG)} P_{(C,\chi)} = 1, \tag{76}$$

$$P_{(C,\chi)} \text{ acts on } V_{(C',\chi')} \text{ by } \delta_{(C,C')}\delta_{(\chi,\chi')}. \tag{77}$$

where we recall $V_{(C,\chi)} = \mathbb{C}[C] \otimes V_\chi$*. If we have such a set of elements, then their corresponding operators* $\{P_{(C,\chi)}(s)\}$ *in* $D(s)$ *form a complete set of orthogonal projectors and hence can be used to construct a measurement. Moreover, the projector* $P_{(C,\chi)}(s)$ *precisely projects states to the irrep* $V_{(C,\chi)}$*. Verify that*

$$P_{(C,\chi)} := \frac{|\chi|}{|Z(r)|}\sum_{c\in C}\sum_{z\in Z(r)} \overline{Tr(\chi(z))} B_c A_{q_c z \overline{q_c}} \tag{78}$$

gives the desired elements ($|\chi|$ *is the dimension of the representation).*

Proof. Recall that the Hilbert space of an excitation (C,χ) in the quantum double model is given by

$$\mathcal{H} = \{|c\rangle \otimes |j\rangle : c \in C, j = 1, ..., |\chi|\} \tag{79}$$

and $D(s)$ acts on $\mathcal{H}$ by

$$B_h |c\rangle \otimes |j\rangle = \delta_{h,c} |c\rangle \otimes |j\rangle \tag{80}$$

$$A_g |c\rangle \otimes |j\rangle = |gc\overline{g}\rangle \otimes \chi(\overline{q_{gc\overline{g}}} g q_c) |j\rangle \tag{81}$$

$$= \sum_i \chi(\overline{q_{gc\overline{g}}} g q_c)_{ij} |gc\overline{g}\rangle \otimes |i\rangle \tag{82}$$

From these it is easy to see that

$$A_g B_h = B_{gh\overline{g}} A_g, B_{h_1} B_{h_2} = \delta_{h_1,h_2} B_{h_2}, A_{g_1} A_{g_2} = A_{g_1 g_2} \tag{83}$$

By Schur Orthogonality, we have

$$\sum_{z\in Z(r)} \overline{\chi(z)}_{nm} \chi'(z)_{n'm'} = \delta_{\chi,\chi'}\delta_{n,n'}\delta_{m,m'} \frac{|Z(r)|}{|\chi|} \tag{84}$$

and since $\overline{Tr(\chi(z))} = \sum_i \overline{\chi(x)_{ii}}$, we can rewrite our expression for $P_{(C,\chi)}$. We have, for $|j\rangle$ a basis vector in some irreducible representation χ',

$$P_{(C,\chi)} |n\rangle \otimes |j\rangle = \frac{|\chi|}{|Z(r)|} \sum_{c \in C} \sum_{z \in Z(r)} \sum_m \overline{\chi(z)}_{mm} B_c \sum_i \chi'(\overline{q_{q_c z \overline{q_c} n q_c \overline{z}\, \overline{q_c}}} q_c z \overline{q_c} q_c)_{ij} |q_c z \overline{q_c} n q_c \overline{z}\, \overline{q_c}\rangle \otimes |i\rangle \tag{85}$$

$$= \frac{|\chi|}{|Z(r)|} \sum_{c \in C} \sum_{z \in Z(r)} \sum_m \overline{\chi(z)}_{mm} \sum_i \chi'(\overline{q_c} q_c z \overline{q_c} q_c)_{ij} \delta_{c, q_c z \overline{q_c} n q_c \overline{z}\, \overline{q_c}} |q_c z \overline{q_c} n q_c \overline{z}\, \overline{q_c}\rangle \otimes |i\rangle \tag{86}$$

$$= \frac{|\chi|}{|Z(r)|} \sum_{c \in C} \sum_m \sum_i \delta_{\chi,\chi'} \delta_{m,i} \delta_{m,j} \frac{|Z(r)|}{|\chi|} |c\rangle \otimes |i\rangle \tag{87}$$

$$= \frac{|\chi|}{|Z(r)|} \sum_{c \in C} \sum_i \delta_{\chi,\chi'} \delta_{i,j} \frac{|Z(r)|}{|\chi|} |c\rangle \otimes |i\rangle \tag{88}$$

$$= \sum_{c \in C} B_c |n\rangle \otimes \delta_{\chi,\chi'} |j\rangle \tag{89}$$

by Schur orthogonality.
Now we check the first property. For some state in $\sum_{c'' \in C'', i \in |\chi''|} v_{c''} |c''\rangle \otimes v_i |i\rangle \in \mathbb{C}[C''] \otimes V_{\chi''}$, we have

$$P_{(C,\chi)} P_{(C',\chi')} v = \sum_{c \in C} B_c \delta_{\chi,\chi''} \sum_{c' \in C'} B_{c'} \delta_{\chi',\chi''} v \tag{90}$$

$$= \sum_{c \in C} \delta_{c,c''} \delta_{\chi,\chi''} \sum_{c' \in C'} \delta_{c',c''} \delta_{\chi',\chi''} v \tag{91}$$

This is only nonzero if $\chi'' = \chi = \chi'$ and $C'' = C = C'$. If this is the case, we have

$$P_{(C,\chi)} P_{(C',\chi')} = \sum_{c \in C} \delta_c (\sum_{c \in C} v_c |c\rangle \otimes ...) \tag{92}$$

$$= \sum_{c \in C} v_c |c\rangle \otimes ... \tag{93}$$

Thus $P_{(C,\chi)} P_{(C',\chi')} = \delta_{C,C'} \delta_{\chi,\chi'}$. However, when we take $\sum_{(C,\chi) \in Irr(DG)} P_{(C,\chi)}$, every irreducible representation is hit and every group element is hit, so it doesn't matter which conjugacy class or representation we have. For $v = \sum_{g \in G} v_g |g\rangle \in$

$\mathbb{C}[G] \otimes \sum_{\chi \in DG} \sum_{x \in |\chi|} v_x |x\rangle \in DG$, we have

$$\sum_{(C,\chi)\in Irr(DG)} P_{(C,\chi)} v = \sum_{C} \sum_{\chi} \sum_{c \in C} \delta_{c,g} \delta_{\chi,\chi'} (\sum_{g} v_g |g\rangle \otimes \sum_{\chi \in DG} \sum_{x \in |\chi'|} v_x |x\rangle) \tag{94}$$

$$= \sum_{g} v_g |g\rangle \otimes \sum_{\chi \in DG} \sum_{x \in |\chi'|} v_x |x\rangle \tag{95}$$

so $\sum_{(C,\chi)\in Irr(DG)} P_{(C,\chi)} = 1$. Lastly, for any element $v = \sum_{c' \in C'} v_{c'} |c'\rangle \otimes \sum_{i \in |\chi'|} v_i |i\rangle \in V_{(C,\chi)}$, we have

$$P_{(C,\chi)} v = \sum_{c \in C} B_c \delta_{\chi,\chi'} \sum_{c' \in C'} v_{c'} |c'\rangle \otimes \sum_{i \in |\chi'|} v_i |i\rangle \tag{96}$$

$$= \sum_{c \in C} \delta_{c,c'} \delta_{\chi,\chi'} \sum_{c' \in C'} v_{c'} |c'\rangle \otimes \sum_{i \in |\chi'|} v_i |i\rangle \tag{97}$$

If $C \neq C'$, c is never c', and this is zero. If $C = C'$, then this is equal to $\delta_{\chi,\chi'} \sum_{c' \in C'} v_{c'} |c'\rangle \otimes \sum_{i \in |\chi'|} v_i |i\rangle$, the identity. Thus $P_{(C,\chi)}$ acts on $V_{(C',\chi')}$ by $\delta_{C,C'} \delta_{\chi,\chi'}$. □

7 Non-abelian Aharonov-Bohm effect

Question. *(Irrep = irreducible representation) We consider two special types of excitations. An anyon of type $(C, \mathbf{1})$ is called a magnetic charge and an anyon of type $(\{e\}, \chi)$ is called an electric charge, where $\mathbf{1}$ means the trivial irrep of the corresponding centralizer and $\{e\}$ is the conjugacy class containing only the identity element. In the latter case, χ is an irrep of G. For a magnetic charge $(C, \mathbf{1})$, a basis for the irrep is given by*

$$\{|c\rangle : c \in C\}, \tag{98}$$

and the action of the double DG is

$$A_g |c\rangle = |gc\overline{g}\rangle \tag{99}$$

$$B_h |c\rangle = \delta_{h,c} |c\rangle . \tag{100}$$

For an electric charge $(\{e\}, \chi)$, a basis for the irrep is given by

$$\{|j\rangle : j = 1, ..., |\chi|\}, \tag{101}$$

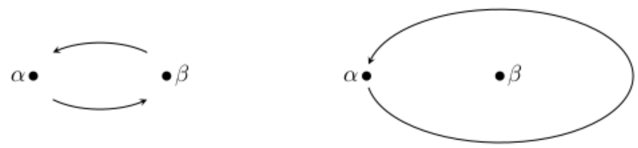

Figure 6: (Left) Swap of α and β in counterclockwise direction. (Right) Drag α around β in counterclockwise direction. This is equivalent to two counterclockwise swaps.

and the action is

$$A_g |j\rangle = \chi(g) |j\rangle \tag{102}$$

$$B_h |j\rangle = \delta_{h,e} |j\rangle . \tag{103}$$

Note that the actions above can all be derived from the general formula on irreps of $\boldsymbol{DG}$*. If we swap an anyon of type* α *with an anyon of type* β *in the counterclockwise direction (see Figure 6 (Left)), then this induces the transformation* $c_{\alpha,\beta}$ *given by:*

$$\alpha \otimes \beta \xrightarrow{R} \alpha \otimes \beta \xrightarrow{Flip} \beta \otimes \alpha, \tag{104}$$

where $\boldsymbol{R} = \sum_g A_g \otimes B_g$*, and the first factor of* $\boldsymbol{R}$ *acts on* α *and the second factor acts on* β*.*

- *If* $\alpha = (\{e\}, \chi), \beta = (C, \boldsymbol{1})$*, a basis for* $\alpha\otimes\beta$ *and* $\beta\otimes\alpha$ *are given, respectively, by*

$$\{|j, c\rangle \, : \, j = 1, ..., |\chi|, c \in C\} \text{ and } \{|c, j\rangle \, : \, j = 1, ..., |\chi|, c \in C\} \tag{105}$$

Write out the transformation $c_{\alpha,\beta}$ *under the bases above. Do the same for* $c_{\beta,\alpha}$*. Swapping* α *and* β *followed by another swap of* β *and* α *is the same as dragging* α *along some closed path around* β *(see Figure 6 (Right)). The net result is a unitary transformation on* $\alpha \otimes \beta$ *given by*

$$\alpha \otimes \beta \xrightarrow{c_{\alpha,\beta}} \beta \otimes \alpha \xrightarrow{c_{\beta,\alpha}} \alpha \otimes \beta. \tag{106}$$

If you have worked out $c_{\alpha,\beta}$ *and* $c_{\beta,\alpha}$*, then you will see that*

$$c_{\beta,\alpha} \circ c_{\alpha\beta} |j, c\rangle = \chi(c) |j\rangle \otimes |c\rangle . \tag{107}$$

This is the non-Abelian Aharonov-Bohm effect for anyons.

- *Work out the formula for $c_{\beta,\alpha} \circ c_{\alpha,\beta}$ in Case I where α, β are two magnetic charges and n Case II where α, β are two electric charges.*

Proof.
- First we act by R.

$$R\,|j,c\rangle = \sum_g A_g\,|j\rangle \otimes B_g\,|c\rangle \tag{108}$$

$$= \sum_g \chi(g)\,|j\rangle \otimes \delta_{g,c}\,|c\rangle \tag{109}$$

$$= \chi(c)\,|j\rangle \otimes |c\rangle \tag{110}$$

$$\xrightarrow{\text{Flip}} |c\rangle \otimes \chi(c)\,|j\rangle \tag{111}$$

Thus $c_{\alpha,\beta}\,|j,c\rangle = |c\rangle \otimes \chi(c)\,|j\rangle$. For $c_{\beta,\alpha}$, we have

$$R\,|c,j\rangle = \sum_g A_g\,|c\rangle \otimes B_g\,|j\rangle \tag{112}$$

$$= \sum_g |gc\overline{g}\rangle \otimes \delta_{g,e}\,|j\rangle \tag{113}$$

$$= |c\rangle \otimes |j\rangle \tag{114}$$

$$\xrightarrow{\text{Flip}} |j\rangle \otimes |c\rangle \tag{115}$$

Thus $c_{\beta,\alpha} \circ c_{\alpha\beta}\,|j,c\rangle = \chi(c)\,|j\rangle \otimes |c\rangle$.

- Case I: $\alpha, \beta = (C, \mathbf{1})$. In this case, a basis for $\alpha \otimes \beta$ is

$$\{|c\rangle \otimes |c'\rangle \,:\, c, c' \in C\} \tag{116}$$

We then have

$$R|c\rangle \otimes |c'\rangle = \sum_g A_g |c\rangle \otimes B_g |c'\rangle \tag{117}$$

$$= \sum_g |gc\overline{g}\rangle \otimes \delta_{g,e} |c'\rangle \tag{118}$$

$$= |c\rangle \otimes |c'\rangle \tag{119}$$

$$\xrightarrow{\text{Flip}} |c'\rangle \otimes |c\rangle \tag{120}$$

$$\xrightarrow{R} \sum_g A_g |c'\rangle \otimes B_g |c\rangle \tag{121}$$

$$= |gc'\overline{g}\rangle \otimes \delta_{g,e} |c\rangle \tag{122}$$

$$= |c'\rangle \otimes |c\rangle \tag{123}$$

$$\xrightarrow{\text{Flip}} |c\rangle \otimes |c'\rangle \tag{124}$$

Thus when magnetic charges are bosons.
Case II: $\alpha, \beta = (\{e\}, \chi)$. In the case, a basis for $\alpha \otimes \beta$ is

$$\{|i\rangle \otimes |j\rangle \,:\, i, j = 1, ..., |\chi|\} \tag{125}$$

We then have

$$R|i\rangle \otimes |j\rangle = \sum_g A_g |i\rangle \otimes B_g |j\rangle \tag{126}$$

$$= \sum_g \chi(g) |i\rangle \otimes \delta_{g,e} |j\rangle \tag{127}$$

$$= \chi(e) |i\rangle \otimes |j\rangle \tag{128}$$

$$= |i\rangle \otimes |j\rangle \tag{129}$$

$$\xrightarrow{\text{Flip}} |j\rangle \otimes |i\rangle \tag{130}$$

$$\xrightarrow{R} \sum_g A_g |j\rangle \otimes B_g |i\rangle \tag{131}$$

$$= \sum_g \chi(g) |j\rangle \otimes \delta_{g,e} |i\rangle \tag{132}$$

$$= |j\rangle \otimes |i\rangle \qquad \xrightarrow{\text{Flip}} |i\rangle \otimes |j\rangle \tag{133}$$

Figure 7: The action of $F^{(i,j)}(\tau)$ for two types of triangles.

Thus electric charges are bosons as well. These results match what we find in toric code.

□

8 Quantum double model for $\mathbb{Z}_2$

Question. *The quantum double based on $G = \mathbb{Z}_2 = \{0, 1\}$ recovers the toric code. In this case, at each edge in the lattice lives a qubit with the standard basis $\{|0\rangle, |1\rangle\}$. There is no need to orient the edges since all group elements are their own inverse and the group is Abelian. Let X and Z be the Pauli matrices.*

- *Work out the formula for the vertex operator $A(v)$ and plaquette operator $B(p)$. These will not be exactly the same as the ones defined originally in toric code, but only differ in a simple way. The two Hamiltonians are equivalent, up to an energy shift.*

- *Let's look at ribbon operators.*

  Let $(i, j) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ be a pair of group elements. If t is a type-I triangle (resp. type-II triangle) (see Figure 7), then $F^{(i,j)}(t)$ acts as $\delta_{j,0}X^i$ (resp. $|j\rangle\langle j|$) on the corresponding edge. The inductive formula for splitting ribbons is given by

$$F^{(i,j)}(t_1 t_2) := \sum_{k \in \mathbb{Z}_2} F^{(i,k)}(t_1) F^{(i,j+k)}(t_2) \tag{134}$$

  Note that arithmetic is performed modulo 2.

  Work out an explicit expression for the ribbon operator $F^{(i,j)}(t)$ where t is shown in Figure 8.

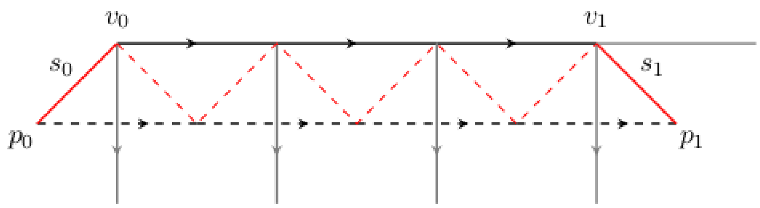

Figure 8: A general ribbon t.

- *To continue, we need to study irreps of $D\mathbb{Z}_2$. Each element of $\mathbb{Z}_2$ represents a conjugacy class, and the centralizer is always $\mathbb{Z}_2$ itself since the group is abelian. An irrep of $\mathbb{Z}_2$ is 1-dimensional and is given by a group element, 0 or 1, corresponding to the trivial and non-trivial irrep. To avoid confusion, let's denote them by [0] and [1]. The [0] irrep maps everything to 1 and the [1] irrep maps a group element i to $(-1)^i$. Therefore, irreps of $D\mathbb{Z}_2$ correspond to*

$$\{(i,[j]) : i,j \in \mathbb{Z}_2\} \tag{135}$$

All of them are 1-dimensional. Show that the matrix element of $D_{(k,l)} = B_k A_l$ in the irrep $(i,[j])$ is given by

$$\Gamma_{11}^{(i,[j])}(D_{(k,l)}) = \delta_{k,i}(-1)^{jl}. \tag{136}$$

- *In the general case, the ribbon operator in the representation basis is given by*

$$F^{(C,\chi;u,u')}(t) = \frac{|(C,\chi)|}{|G|}\sum_{h,g}\Gamma_{u,u'}^{(C,\chi)}(D_{(h,g)})F^{(h,g)}(t). \tag{137}$$

In our case, this formula can be simplified as

$$F^{(i,[j])}(t) = \frac{1}{2}\sum_{l=0}^{1}(-1)^{jl}F^{(i,l)}(t). \tag{138}$$

What is the explicit formula of $F^{(i,[j])}(t)$ for the ribbon t in Figure 8?

Proof. • For a general G, with a lattice $\mathscr{L}$ with set of edges E, we have

$$A(v) := \frac{1}{|G|} \sum_{g \in G} A_g(v), B(p) = B_e(v,p) \tag{139}$$

For $G = \mathbb{Z}_2$, this becomes

$$A(v) = \frac{1}{2}(Id + (\otimes_{e \in star(v)} X_e) \otimes (\otimes_{e \in E - star(v)} Id_e)), \tag{140}$$

$$B(p) = \frac{1}{2}(Id + (\otimes_{e \in \partial p} Z_e) \otimes (\otimes_{e \in E - \partial p} Id_e)), \tag{141}$$

for if p is a plaquette with edges $p_i \in \mathbb{Z}_2, i = 1, ..., 4$, if $p_1 p_2 p_3 p_4 = 1$, $B(p)$ sends the state to zero, otherwise acts by the identity.

- There are 7 triangles in this ribbon operator, and we label them t_i in the order they appear from the left, crossing edges e_i labelled in the same way. Let k_i be the summed index in the inductive split of the ribbon operator when we separate out the i^{th} triangle from the ribbon. For example, we use k_7 and k_6 in

$$F^{(i,j)}(t_1t_2t_3t_4t_5t_6t_7) = \sum_{k_7 \in \mathbb{Z}_2} F^{(i,k_7)}(t_1t_2t_3t_4t_5t_6) F^{(i,j+k_7)}(t_7) \tag{142}$$

$$= \sum_{k_7 \in \mathbb{Z}_2} F^{(i,k_7)}(t_1t_2t_3t_4t_5t_6)[\delta_{j+k_7,0} X^i]_{e_7} \tag{143}$$

$$= \sum_{k_7 \in \mathbb{Z}_2} \sum_{k_6 \in \mathbb{Z}_2} F^{(i,k_6)}(t_1t_2t_3t_4t_5) F^{(i,k_7+k_6}(t_6)[\delta_{j+k_7,0} X^i]_{e_7} \tag{144}$$

$$= \sum_{k_7 \in \mathbb{Z}_2} \sum_{k_6 \in \mathbb{Z}_2} F^{(i,k_6)}(t_1t_2t_3t_4t_5)[|k_7 + k_6\rangle \langle k_7 + k_6|]_{e_6} [\delta_{j+k_7,0} X^i]_{e_7} \tag{145}$$

$$\vdots \tag{146}$$

Following this through, we eventually get

$$\sum_{k_2,k_3,k_4,k_5,k_6,k_7 \in \mathbb{Z}_2} [\delta_{k_2,0} X^i]_{e_1} [|k_3 + k_2\rangle \langle k_3 + k_2|]_{e_2} [\delta_{k_4+k_3,0} X^i]_{e_3} [|k_5 + k_4\rangle \tag{147}$$

$$\times \langle k_5 + k_4|]_{e_4} [\delta_{k_6+k_5,0} X^i]_{e_5} [|k_7 + k_6\rangle \langle k_7 + k_6|]_{e_6} [\delta_{j+k_7,0} X^i]_{e_7} \tag{148}$$

which, after summing, we have

$$[X^i] \otimes [2\,|0\rangle\langle 0| + 2\,|1\rangle\langle 1|] \otimes [2X^i] \otimes [2\,|0\rangle\langle 0| + 2\,|1\rangle\langle 1|] \tag{149}$$

$$\otimes[2X^i] \otimes [2\,|0\rangle\langle 0| + 2\,|1\rangle\langle 1|] \otimes [\delta_{j,0}X^i + \delta_{j+1,0}X^i] \tag{150}$$

$$= [X^i] \otimes [2Id] \otimes [2X^i] \otimes [2Id] \tag{151}$$

$$\otimes[2X^i] \otimes [2Id] \otimes [X^i] \tag{152}$$

- With this usual quantum double operators

$$B_k\,|n,[n']\rangle = \delta_{k,n}\,|n,[n']\rangle\,,\, A_l\,|n,[n']\rangle = \sum_m \Gamma^{\chi}_{mn'}(\overline{q_{ln\bar{l}}}lq_n)\,|ln\bar{l},[m]\rangle \tag{153}$$

In abelian $\mathbb{Z}_2$ with $\chi = (i,[j])$, we get

$$D_{(k,l)}\,|n,[n']\rangle = B_k A_l\,|n,[n']\rangle \tag{154}$$

$$= B_k \sum_m \Gamma^{(i,[j])}_{mn'}(l)\,|n,[m]\rangle \tag{155}$$

$$= \delta_{k,i} \sum_m \Gamma^{(i,[j])}_{mn'}(l)\,|n,[m]\rangle \tag{156}$$

so $[D_{(k,l)}]_{11} = \delta_{k,i}\Gamma^{(i,[j])}_{11}(l)$. If $j = 0$ everything is mapped to 1. If $j = 1$, we map to $(-1)^l$. Thus the matrix element corresponds to $\delta_{k,i}(-1)^{jl}$.

- We showed above that $F^{(i,l)}(t) = 2\,|0\rangle\langle 0| + 2\,|1\rangle\langle 1| = 2Id$ on edges, $2X^i$ on edges crossed by the τ triangles, and X^i on the ends. Thus we have

$$[\frac{1}{2}X^i] \otimes [Id] \otimes [X^i] \otimes [Id] \tag{157}$$

$$\otimes[X^i] \otimes [Id] \otimes [\frac{1}{2}X^i] \tag{158}$$

$$+(-1)^j[\frac{1}{2}X^i] \otimes [Id] \otimes [X^i] \otimes [Id] \tag{159}$$

$$\otimes[X^i] \otimes [Id] \otimes [\frac{1}{2}X^i] \tag{160}$$

On edges crossed by τ, this is the operation by Z, and on edges on the ribbon itself, this is the operation by X if $i = 1$.These are the string operators from toric code.

□