Quantum Groups

In the spring of 2019 Professor Daniel Bump gave a series of lectures on Quantum Groups (MATH 263C). These notes are 'live-texed' by Alec Lau (with a few embellishments here and there), and thus all typos are to be reported to Alec Lau at aszlau@gmail.com.

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1 Bialgebras and Hopf Algebras

1.1 Braids, Categories, & Braided Categories

The braid group B_n of n braids is generated by braids $t_1, ..., t_{n-1}$; braids are multiplied by concatenating. The braid relations are

$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \tag{1}$$

$$t_i t_j = t_j t_i \text{ if } |i-j| \geq 2 \tag{2}$$

Strands can only move in one direction, and thus knots like the Trefoil knot cannot be modeled by the braid group.

Let U, V, W be vector spaces. We need $R \in End(U \otimes V), S \in End(U \otimes W), T \in End(V \otimes W)$. This version of the Yang-Baxter equation is the identity $(T \otimes I_U)(I_V \otimes S)(R \otimes I_W) = (I_W \otimes U)(S \otimes I_V)(I_U \otimes T)$.

A *monoidal category* is a category with a bifunctor $\otimes$ that is associative i.e. natural isomorphisms $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ such that all identities satisfy the *Pentagon Identity*:

$$\begin{array}{ccccc} & & (A \otimes B) \otimes (C \otimes D) & & \\ & \nearrow & & \searrow & \\ ((A \otimes B) \otimes C) \otimes D & & & & A \otimes (B \otimes (C \otimes D)) \\ \downarrow & & & & \uparrow \\ (A \otimes (B \otimes C)) \otimes D & & \longrightarrow & & A \otimes ((B \otimes C) \otimes D) \end{array}$$

Example 1. *The category of sets with $\otimes = \times$, with unit object $I = \{1\}$ the set with one object.*

A braided category, to be defined later, is a monoidal category with a braiding $c_{V,W} : V \otimes W \to W \otimes V$.

The category of modules of a Quasitriangular Hopf Algebra is braided. Hence Quantum Groups $\Rightarrow$ Braided categories $\Rightarrow$ Applications, such as solvable lattice models and knot invariants.

1.2 Monoids (Algebras) from Categories of Sets (Vector Spaces)

A quantum group should be a deformation of a group G, typically a Lie group. The quantum group depends on a parameter q, wherein the classical limit $q \mapsto 1$ the group is recovered. However, groups are rigid and cannot be deformed this way. The idea is to replace the group by another object with the same representation theory of G which lives in a category that *does* allow deformations. This is the category of *Hopf Algebras*.

A monoid in the category of vector spaces can be an *algebra*.

A monoid in the category of vector spaces can be a *bialgebra*.

A group in the category of vector spaces can be a *Hopf Algebra*.

The category of sets in monoidal with unit element $I = \{1\}$ and monoidal operation $\times$. Let $\eta : I \to M$ map $1 \mapsto 1_M$ and $\mu : M \times M \to M$ be the multiplication in the monoid. **A monoid can be defined in terms of these in the category of sets, and similarly an alegebra can be defined in terms of these in the category of vector spaces**:

$$\begin{array}{ccc} M\times M\times M & \xrightarrow{\mu\times 1} & M\times M \\ \big\downarrow{\scriptstyle 1\times\mu} & & \big\downarrow{\scriptstyle \mu} \\ M\times M & \xrightarrow{\quad\mu\quad} & M \end{array} \qquad \begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{\mu\otimes 1} & A\otimes A \\ \big\downarrow{\scriptstyle 1\otimes\mu} & & \big\downarrow{\scriptstyle \mu} \\ A\otimes A & \xrightarrow{\quad\mu\quad} & A \end{array}$$

$$m\cdot 1 = m \Rightarrow \begin{array}{ccc} M\times M & & \\ \big\uparrow{\scriptstyle 1\times\eta} & \searrow{\scriptstyle\mu} & \\ M\times Id & \xrightarrow{\cong} & M \end{array} \qquad \begin{array}{ccc} M\times M & & \\ {\scriptstyle \eta\times 1}\big\uparrow & \searrow{\scriptstyle\mu} & \\ Id\times M & \xrightarrow{\cong} & M \end{array} \Leftarrow 1\cdot m = m$$

Define a multiplication on A by $a \cdot b = \mu(a \otimes b)$, and replace the 3 commutative diagrams above with $M \mapsto A$, $\times \mapsto \otimes$, the first shown above. The diagrams thus mean $(ab)c = a(bc)$, so A is a ring with identity $1 = \eta(1_K)$, with η embedding K in the center of A. Thus A is an algebra.

We can improve the description of a monoid in terms of categories via the diagonal map $\Delta : M \to M \times M$:

$$\begin{array}{ccc} M & \xrightarrow{\quad\Delta\quad} & M\times M \\ \big\downarrow{\scriptstyle\Delta} & & \big\downarrow{\scriptstyle\Delta\times 1} \\ M\times M & \xrightarrow{1\times\Delta} & M\times M\times M \end{array}$$

$$\begin{array}{ccc} M\times M & & \\ \big\downarrow{\scriptstyle 1\times\epsilon} & \nwarrow{\scriptstyle\Delta} & \\ M\times Id & \xleftarrow{\cong} & M \end{array} \qquad \begin{array}{ccc} M\times M & & \\ {\scriptstyle \epsilon\times 1}\big\downarrow & \nwarrow{\scriptstyle\Delta} & \\ Id\times M & \xrightarrow{\cong} & M \end{array}$$

To further develop Δ, we add the τ "flip" map: $\tau : M \times M \to M \times M, \tau((x, y)) = (y, x)$:

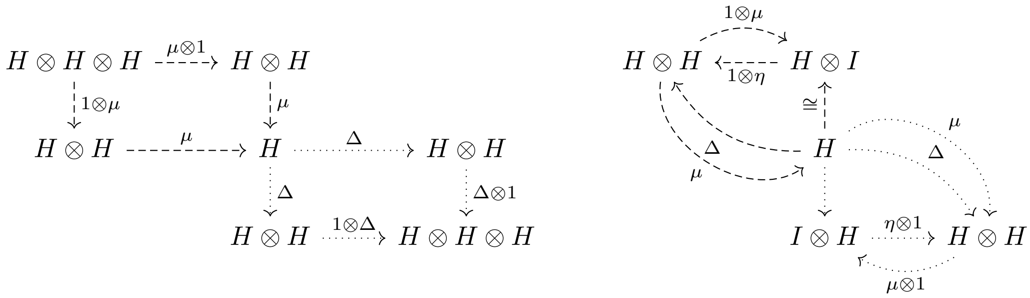
$$\begin{array}{ccc} M\times M & \xrightarrow{\Delta\times\Delta} & M\times M\times M\times M \\ \big\downarrow{\scriptstyle\mu} & & \big\downarrow{\scriptstyle 1\times\tau\times 1} \\ M & & M\times M\times M\times M \\ & \searrow{\scriptstyle\Delta} & \big\downarrow{\scriptstyle\mu\times\mu} \\ & & M\times M \end{array}$$

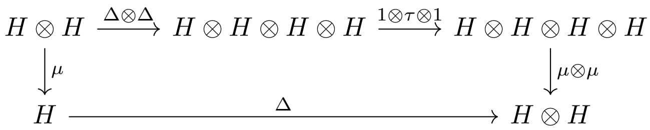
$$(x, y) \mapsto (xy, xy)$$

$$\begin{array}{ccc} M\times M & \underset{\epsilon\times\epsilon}{\overset{\eta\times\eta}{\rightleftarrows}} & I\times I \\ {\scriptstyle\Delta}\big\uparrow & & {\scriptstyle\cong}\big\uparrow \\ M & \underset{\eta}{\overset{\epsilon}{\rightleftarrows}} & I \end{array}$$

Dualizing the vector space in the same way, we replace our mondoid with a vector space and $\times$ with $\otimes$.

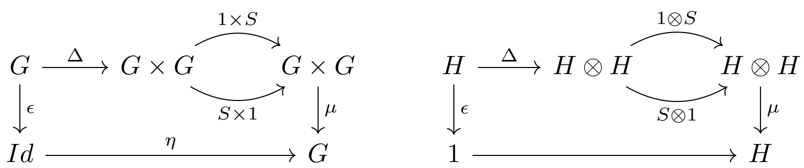
We put all of our maps together to define a **bialgebra** via the category of vector spaces: Let H be a vector space. Then we have the following commutative diagrams:





Where the dashed lines represent associativity and unit maps, respectively, and the dotted lines represent coassociativity and counit maps, respectively. The third diagram is called the *Hopf Axiom.*

A group in the category of vector spaces is a Hopf Algebra. Let G be a group, and let $S : G \to G$ be the map $S(g) = g^{-1}$. We need to express the axiom $gg^{-1} = g^{-1}g = 1$ in terms of maps. We can use the diagonal map, the multiplication map, and the unit/counit maps. So a Hopf algebra is a bialgebra H with a linear map $S : H \to H$ satisfying the additional (self-dual) axioms:



This axiom is self-dual, so reversing all arrows in the definition doesn't change the Hopf axiom.

1.3 Hopf Algebras from a Group G

There are two ways of obtaining a Hopf algebra from a finite group G.

The group algebra $K[G]$ has comultiplication $\Delta(g) = g \otimes g$ on a basis element $g \in G$ and is a Hopf algebra.

Dually, the ring $\mathcal{O}(G)$ of functions on G is a Hopf algebra. The multiplication is pointwise, so the algebra is commutative. The comultiplication is given by identifying $\mathcal{O}(G\times G) = \mathcal{O}(G)\otimes\mathcal{O}(G)$, $\Delta f((g, h)) = f(gh)$.

We let G be an affine algebraic group over $\mathbb{C}$, and so $G(\mathbb{C})$ of complex points of G is a Lie Group. Call its Lie Algebra $\mathfrak{g}$. The *Universal Enveloping Algebra* $U(\mathfrak{g})$ is an associative ring with $\mathfrak{g}$ as a vector subspace. We have

$$[X, Y] = XY - YX \tag{3}$$

for $X, Y \in \mathfrak{g}$, the multiplication is in $U(\mathfrak{g})$. This is called universal because if A is any associative algebra, and $\rho : \mathfrak{g} \to A$ a linear map where $\rho([X,Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$, then $\rho : U \to A$ extends to a homomorphism. Elements of $\mathfrak{g}$ can be thought of as differential operators or distributions on G concatenated at the identity. If $X \in G$,

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \epsilon(X) = 0, S(X) = -X \tag{4}$$

Example 2. *The Lie Algebra of $SL_2(\mathbb{C})$ is the set of matrices with trace 0. It is a 3-dimensional vector space with generators*

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The bracket operations are

$$[X,Y] = H, [H,X] = 2X, [H,Y] = -2Y$$

We have to be careful, though, because the multiplication here is not matrix multiplication but multiplication in U.

The other way to associate a Hopf algebra with a group is the ring $\mathcal{O} = \mathcal{O}(G(\mathbb{C}))$ of rational (polynomial) functions. Unlike the enveloping algebra, this is a commutative algebra, and the multiplication is encoded in the comultiplication. Let V be an affine algebraic variety and $\mathcal{O}(V)$ its algebra, we can identify

$$\mathcal{O}(V \times V) \cong \mathcal{O}(V) \otimes \mathcal{O}(V) \tag{5}$$

where thus the multiplication morphism $M \times M \to M$ corresponds to an algebra homomorphism $\Delta\mathcal{O} \to \mathcal{O} \otimes \mathcal{O}$.

1.4 $U(\mathfrak{g})$ vs. $\mathcal{O}(G(\mathbb{C}))$

Both $U(G)$ and $\mathcal{O}(G)$ have deformations that involve q, and the $q \to 1$ limit yields the non-deformed theory, and $q \to 0$ yields the theory of crystal bases, a rabbit hole all on its own. The case $G = SL(2)$ is enough for significant applications like the Jones polynomial and the six-vertex model in statmech.

$U(\mathfrak{g})$ is cocommutative, and the multiplication encodes the multiplication of the group.

Remark 1. *Cocommutative implies $\tau \circ \Delta = \Delta$, where τ is the 'flip' map.*

The $\mathcal{O}(G(\mathbb{C}))$ funtion algebra is commutative, with comultiplication encoding the multiplication of the group.

2 Sweedler Notation and Category Theory

2.1 Sweedler's Notation

Using ordinary ring notation for the multiplication and unit μ and η. Now we can say, for $a, b \in H$, $a \cdot b = \mu(a \otimes b), 1_H = \eta(1_K)$. We can use η to identify K with a subring in the center of H.

Using **Sweedler's Notation**, we have

$$\Delta(a) = \sum_{i=1}^{N} a_i' \otimes a_i'' := \Delta(a) = a_{(1)} \otimes a_{(2)} \tag{6}$$

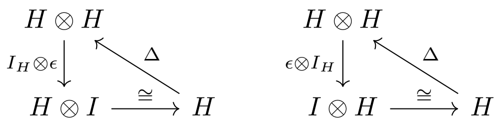
Coassociativity gives us

$$(\Delta \otimes 1_H)\Delta(a) = (1_H \otimes \Delta)\Delta(a) \tag{7}$$

$$a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} = a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)} \tag{8}$$

$$= a_{(1)} \otimes a_{(2)} \otimes a_{(3)} \tag{9}$$

Rewriting the counit axiom in Sweedler's notation,



$$a = a_{(1)}\epsilon(a_{(2)}) = \epsilon(a_{(1)})a(2) \tag{10}$$

In the same way, we can write

$$\epsilon(a) = a_{(1)}S(a_{(2)}) = S(a_{(1)})a_{(2)} \tag{11}$$

If A, B are algebras, so is $A \otimes B$ with multiplication $(a \otimes b)(a' \otimes b') = (aa' \otimes bb'), \mu_{A\otimes B} = (\mu_A \otimes \mu_B)(1_A \otimes \tau \otimes 1_B)$. Also, $(xy)_{(1)} \otimes (xy)_{(2)} = x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)}$

2.2 S(xy)=S(y)S(x)

Lemma 1. $S(ab) = S(b)S(a)$

Proof. This is similar to the proof that $(xy)^{-1} = y^{-1}x^{-1}$: $(xy)^{-1} = (xy)^{-1}xyy^{-1}x^{-1} = y^{-1}x^{-1}$. Take the Hopf analogue of $(xy)^{-1} = (xy)^{-1}xyy^{-1}x^{-1} : S(x_{(1)}y_{(1)})x_{(2)}y_{(2)}S(y_{(3)})S(x_{(3)})$. The idea is to proof that his analogue is equal to both $S(xy)$ and $S(y)S(x)$. Also note that ϵ is a scalar valued ring homomorphism, and thus commutes nicely and has the property $\epsilon(a)\epsilon(b) = \epsilon(ab)$.

$$
\begin{aligned}
&S(x_{(1)}y_{(1)})x_{(2)}y_{(2)}S(y_{(3)})S(x_{(3)}) && (12)\\
&= S(x_{(1)}y_{(1)})x_{(2)}\epsilon(y_{(2)})S(x_{(3)}) && (13)\\
&= S(x_{(1)}y_{(1)})x_{(2)}S(x_{(3)})\epsilon(y_{(2)}) && (14)\\
&= S(x_{(1)}y_{(1)})\epsilon(x_{(2)})\epsilon(y_{(2)}) && (15)\\
&= S(x_{(1)}y_{(1)})\epsilon(x_{(2)}y_{(2)}) && (16)\\
&= S(x_{(1)}y_{(1)}\epsilon(x_{(2)}y_{(2)})) && (17)\\
&= S((xy)_{(1)}\epsilon((xy)_{(2)})) && (18)\\
&= S(xy) && (19)
\end{aligned}
$$

where (13,15) use the counit axiom, (16) uses the fact that ϵ is a homomorphism, (17) uses the fact that ϵ is scalar-valued, and (18) uses the Hopf Axiom:$(xy)_{(1)} \otimes (xy)_{(2)} = x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)}$. We can also do

$$
\begin{aligned}
&S(x_{(1)}y_{(1)})x_{(2)}y_{(2)}S(y_{(3)})S(x_{(3)}) && (20)\\
&= S(x_{(1)(1)}y_{(1)(1)})x_{(1)(2)}y_{(2)(2)}S(y_{(2)})S(x_{(2)}) && (21)\\
&= S((x_{(1)}y_{(1)})_{(1)})(x_{(1)}y_{(1)})_{(2)}S(y_{(2)})S(x_{(2)}) && (22)\\
&= \epsilon(x_{(1)}y_{(1)})S(y_{(2)})S(x_{(2)}) && (23)\\
&= \epsilon(x_{(1)})\epsilon(y_{(1)})S(y_{(2)})S(x_{(2)}) && (24)\\
&= S(\epsilon(y_{(1)})y_{(2)})S(\epsilon(x_{(1)})x_{(2)}) && (25)\\
&= S(y)S(x) && (26)
\end{aligned}
$$

where (29) is due to (8,9) above, (22) is the Hopf Axiom, (23) is the counit axiom, (24) is the fact that ϵ is a homomorphism, and (25) is the scalar value. Basically, we took $S(x_{(1)}y_{(1)})x_{(2)}y_{(2)}S(y_{(3)})S(x_{(3)})$ and collapsed the first 3 factors and the last 4 factors, the same as in the proof $(xy)^{-1} = y^{-1}x^{-1}$. □

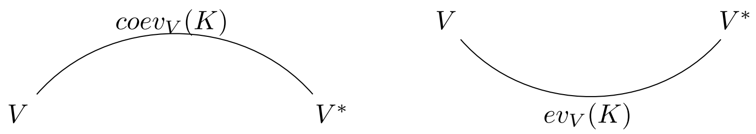
2.3 Monoidal Categories

Modules over a bialgebra are a monoidal category: if V, W are modules over H, then $V \otimes W$ is a module over $H \otimes H$, due to comultiplication. Δ is an $H-$homomorphism, we can apply our module structure to $H : a(v \otimes w) = a_{(1)}v \otimes a_{(2)}w$. Due to coassociativity, $U \otimes (V \otimes W)$ has the same structure as $(U \otimes V) \otimes W$.

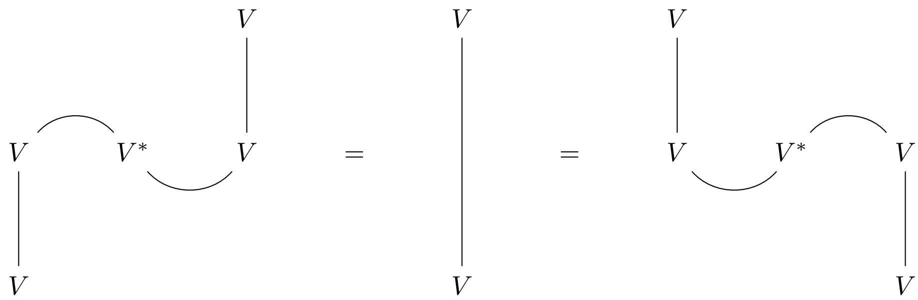
Definition 1. *A* ***rigid category*** *is a category where the objects have duals. Let V be an object in* *a monoidal category with unit object I. We have morphisms $ev_V : V^* \otimes V \to I, coev_V : I \to V \otimes V^*$ called evaluation and coevaluation, respectively, that abstract the notion of a right dual. A left dual functions the same way. These maps have the following axioms:*

$$(1_V \otimes ev_V) \circ (coev_V \otimes 1_V) = 1_V, (ev_V \otimes 1_{V^*}) \circ (1_{V^*} \otimes coev_V) = 1_{V^*} \tag{27}$$

Finite-dimensional modules over a Hopf Algebra are a rigid monoidal category. Let $I = K$, V^* =Hom(V, K), and $ev_V(v^* \otimes v) = v^*(v)$, evaluating the linear functional $v^* \in V^*$ at v. It now remains to define the coevaluation map $K \to V \otimes V^*$. Let v_i be a basis of V and v_i^* be a basis of V^*. Define $coev_V(a) = a \sum v_i \otimes v_i^*$. Once can visualize this as the following diagram, read top to bottom:

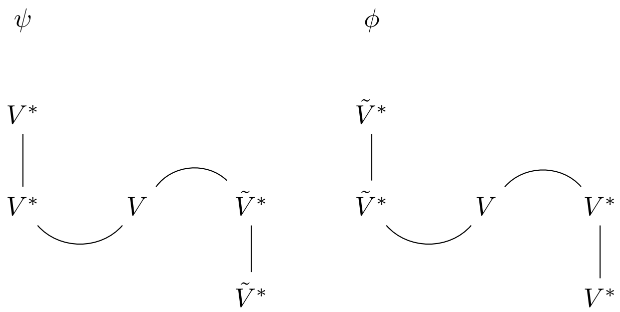


We can further visualize the $(1_V \otimes ev_V) \circ (coev_V \otimes 1_V) = 1_V, (ev_V \otimes 1_{V^*}) \circ (1_{V^*} \otimes coev_V) = 1_{V^*}$ axiom by

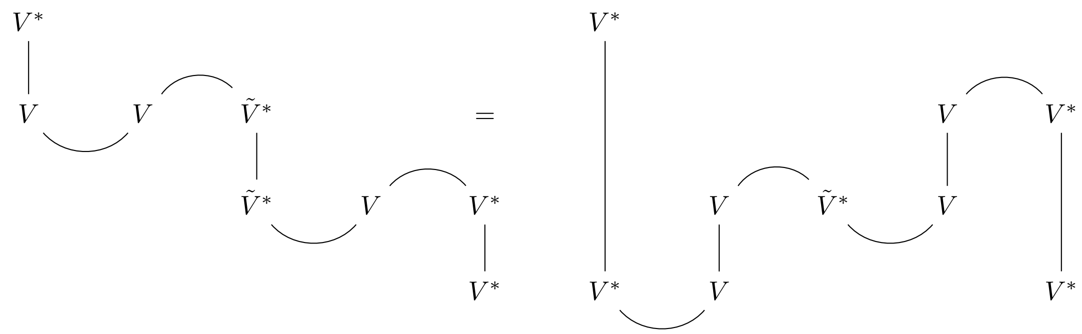


2.4 Uniqueness of the Dual

Now we prove the uniqueness of the dual V^*. Suppose we have another dual $\tilde{V}^*$ with (co)evaluation maps $\tilde{ev}_V, \tilde{coev}_V$, and maps $\psi : \tilde{V}^* \to V^*, \psi = (\tilde{ev}_V \otimes I_{V*})(I_{\tilde{V}^*} \otimes coev_V)$, $\phi : V^* \to \tilde{V}^*, \phi = (ev_V \otimes I_{\tilde{V}^*})(I_{V^*} \otimes \tilde{coev}_V)$:



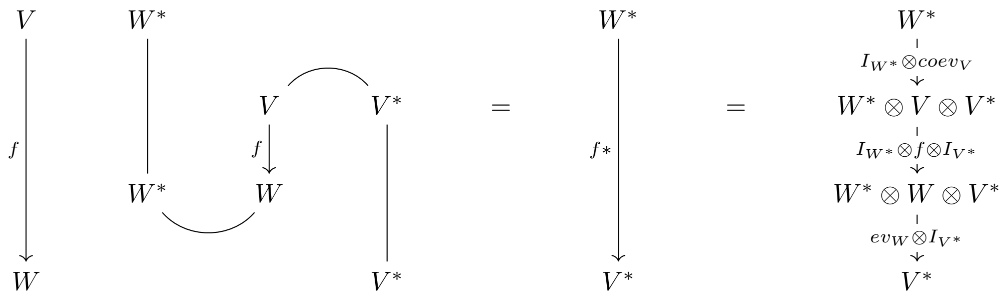
We can compose ψ and ϕ to get some insights: $\psi \circ \phi = (ev_V \otimes I_{\tilde{V}^*})(I_{V^*} \otimes \tilde{coev}_V)(\tilde{ev}_V \otimes I_{V*})(I_{\tilde{V}^*} \otimes coev_V) = (ev_V \otimes I_{V^*})(I_{V^*} \otimes I_V \otimes \tilde{ev}_V \otimes I_{V^*})(I_{V^*} \otimes \tilde{coev}_V \otimes I_V \otimes I_{V^*})(I_{V^*} \otimes coev_V)$:



Switch $\tilde{ev}_V$ and $\tilde{coev}_V$, and, using the fact that $\otimes$ is a bifunctor, we use the axioms twice to conclude $\psi \circ \phi = I_{V^*}$ and $\phi \circ \psi = I_{\tilde{V}^*}$. Thus ψ and ϕ are inverse isomorphisms, proving that the dual is unique.

2.5 Rigid Categories

Recall that a rigid category is a category where every object has a dual. Suppose that V, W are objects in a rigid category and $f : V \to W$ is a morphism. Define a morphism $f^* : W^* \to V^*$ by $f^* = (ev_W \otimes 1_{V^*})(1_{W^*} \otimes f \otimes 1_{V^*})(I_{W^*} \otimes coev_V)$:



3 Braided categories, quasitriangular Hopf algebras, quantized enveloping algebras, and Reidemeister moves.

For $\mathcal{C}$ a monoidal category, we have natural isomorphisms $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ for A, B, C objects in $\mathcal{C}$. Denote either as $A \otimes B \otimes C$. This identification doesn't cause any problems, as shown by MacLane's Coherence Theorem. In a *braided category* there are braidings $c_{A,B} : A \otimes B \to B \otimes A$ where $c_{A,B} \circ c_{B,A}$ is not necessarily the identity. To see this, look at an actual braiding of two strands. This will create a twist. I.e. $c_{A,B}, c_{B,A}^{-1}$ are distinct isomorphisms. These associativity conditions are natural, i.e.

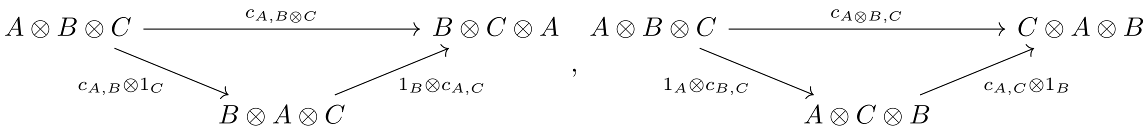
$$\begin{array}{ccc} (A \otimes B) \otimes C & \xrightarrow{\cong} & A \otimes (B \otimes C) \\ {\scriptstyle (\alpha\otimes\beta)\otimes\gamma} \downarrow & & \downarrow {\scriptstyle \alpha\otimes(\beta\otimes\gamma)} \\ (A' \otimes B') \otimes C' & \xrightarrow{\cong} & A' \otimes (B' \otimes C') \end{array}$$

The first axiom of a braided category is the braidings have to be natural, i.e. for $\alpha : A \to$

$A', \beta : B \to B'$,

$$(\beta \otimes \alpha) \circ c_{A,B} = c_{A',B'} \circ (\alpha \otimes \beta) \tag{28}$$

The $c_{A,B}$ morphism is sometimes called the **R-matrix**. Assume it satisfies



The Yang-Baxter Equation is true in a braided monoidal category. Visualize the braidings of 3 strands to convince yourself that this is true.

We have the slogan **modules over a quasitriangular bialgebra are a braided category**. A quasitriangular/braided Hopf Algebra was introduced by Drinfeld in his 1986 ICM lecture.

3.1 Quasitriangularity

Let $R \in H \otimes H$ be an invertible element, and U, V be $H-$modules with the flip map τ.

Lemma 2. *Suppose we have $x \in H$ such that $R\Delta(x)R^{-1} = \tau\Delta(x)$. Then for all U, V, $c_{u\otimes v} = \tau R(u \otimes v)$ is an $H-$module homomorphism.*

Proof. $R\Delta(x)R^{-1} = \tau\Delta(x) \Rightarrow R\Delta(x) = \tau\Delta(x)R$. Denote $R = R^{(1)} \otimes R^{(2)}$. Using Sweedler's Notation,

$$R^{(1)}x_{(1)} \otimes R^{(2)}x_{(2)} = x_{(2)}R^{(1)} \otimes x_{(1)}R^{(2)} \tag{29}$$

. Now suppose $u \otimes v \in U \otimes V$. We want to show $\tau R(x(u \otimes v)) = x\tau R(u \otimes v)$. Apply the map $a \otimes b \mapsto bv \otimes au$ in (29) and we are done. □

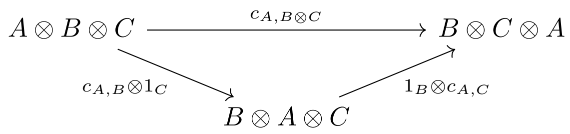
Denote $R_{12} = R^{(1)} \otimes R^{(2)} \otimes 1_R, R_{13} = R^{(1)} \otimes 1_R \otimes R^{(2)}, R_{12} = 1_R \otimes R^{(1)} \otimes R^{(2)}$.

Definition 2. *A **quasitriangular Hopf Algebra** is a Hopf Algebra with an $R \in H \otimes H$ such that*

$$R\Delta(x)R^{-1} = \tau\Delta, (\Delta \otimes 1)R = R_{13}R_{23}, (1 \otimes \Delta)R = R_{13}R_{12} \tag{30}$$

*called the **universal R-matrix**.*

Lemma 3. *The axiom $(1 \otimes \Delta)R = R_{13}R_{12}$ is equivalent to the axiom*



Proof. Define the map $\theta(a \otimes b \otimes c) = b \otimes c \otimes a$. First we argue that the top arrow is

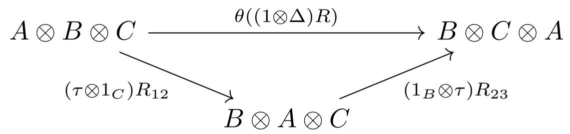
$$c_{A,B\otimes C}(a, b, c) = \theta((1 \otimes \Delta)R)(a \otimes b \otimes c) \tag{31}$$

. Treat $d := a \otimes b$ as a unit. Consider $c_{A,B\otimes B}(a \otimes d) = \tau R(a \otimes d) = \tau(R^{(1)}a \otimes R^{(2)}d)$, where θ is the τ map in this case. Using the fact that elements of the Hopf Algebra on a tesnor product of modules is through the tensor product, so we get (31) as we recognize $R^{(2)}d = \Delta(R^{(2)})(a \otimes b)$.

Now we want to show that $\theta((1 \otimes \Delta)R) = (1_B \otimes \tau)R_{12}(\tau \otimes 1_C)R_{12}$. We know that $(\tau \otimes 1_C)R_{23}(\tau \otimes 1_C) = R_{13}$ and $(1_B \otimes \tau)(\tau \otimes 1_C) = \theta$. Thus

$$(1_B \otimes \tau)R_{23}(\tau \otimes 1_C)R_{12} = (1_B \otimes \tau)(\tau \otimes 1_C)R_{13}R_{23} = \theta R_{13}R_{23} \tag{32}$$

One axiom for a quasitriangular Hopf algebra is $(1 \otimes \Delta)R = R_{13}R_{23}$. The commutativity of



follows and we have proved an axiom of a braided category. □

The mirror image and naturality are easy and, canonically, "left to the reader."

3.2 An Example

Consider the group $G = \mathbb{Z}^n$. Let $H = \mathbb{C}[G]$ be the group algebra and let $q = e^{2i\pi/n}$(the n^{th} root of unity). It is a Hopf algebra with $\Delta(g^a) = g^a \otimes g^a$. Define $R = \frac{1}{n}\sum_{a,b \mod n}^{-ab} g^a \otimes g^b \in H \otimes H$.

Lemma 4. $R^{-1} = \frac{1}{n}\sum_{a,b \mod n} q^{ab}g^a \otimes g^b$.

Proof. The product of the elements is $\frac{1}{n^2}\sum_{a,b,c,d \mod n} q^{-ab+cd}g^{a+c} \otimes g^{b+d} =$ $\frac{1}{n^2}\sum_{t,u}(\sum_{a,b} a^{-ab+(t-a)(u-b)})g^t \otimes g^u$. The inner sum is 0, or n^2 if $t = u = 0$. □

Theorem 1. *H is quasitriangular with universal R-matrix R.*

Proof. $R\Delta R^{-1} = \tau\Delta$ is trivial due to cocommutativity and commutativity. $R_{13}R_{12} = \frac{1}{n^2}(\sum_{a,b} q^{-ab}g^a \otimes 1 \otimes g^b)(\sum_{c,d} q^{-cd}g^c \otimes g^d \otimes 1) = \frac{1}{n^2}\sum_{a,b,c,d} q^{-ab-cd}g^{a+c} \otimes g^d \otimes g^b = \frac{1}{n^2}\sum_{t,b,d}(\sum_a q^{-ab-(t-a)d})g^t \otimes g^d \otimes g^b$. The sum is 0 unless $b = d$. Thus this becomes $\frac{1}{n}\sum_{t,b}(\sum_a q^{-tb})g^t \otimes g^b \otimes g^b = (1 \otimes \Delta)R$ □

This proves $R_{13}R_{12} = (1 \otimes \Delta)R$, and $R_{13}R_{23} = (\Delta \otimes 1)R$ is the same deal.

3.3 Quantized Enveloping Algebras

There are two kinds of Hopf algebras that can arise from a Lie Group: The affine algebra/coordinate ring and the enveloping algebra, and both have $q-$(quantum) deformations. If $\mathfrak{g}$ is a semisimple Lie algebra, and $H = U_g(\mathfrak{g})$ is the quantized enveloping algebra, R lives in a *completion* of H. Thus, H isn't a quasitriangular Hopf algebra. If q is a root of unity then $U_q(\mathfrak{g})$ has a finite-dimensional quotient that *is* a QHA.

3.4 Example

Recall the Lie Algebra $\mathfrak{sl}_2$ has basis $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, with $[E, F] = H, [H, E] = 2E, [H, F] = -2F$. To construct the quantized algebra we replace H with an element that can be thought of $K := \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$, such that K yields $KEK^{-1} = q^2 E, KFK^{-1} = q^{-2}F, EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$. This defines $U_q(\mathfrak{sl}_2)$.

Remark 2. *It isn't possible to set $q = 1$ to recover $U(\mathfrak{sl}_2$, but recovering can be done. See Kassel's Quantum Groups.*

There is an algebra homomorphism $\Delta : H \to H \otimes H$ such that $\Delta(K) = K \otimes K, \Delta E = E \otimes K + 1 \otimes E, \Delta(F) = F \otimes 1 + K^{-1} \otimes F$. These images indeed satisfy the relations, so this is a homomorphism. Define a counit ϵ to $\mathbb{C}$ where $\epsilon(K) = 1, \epsilon(E) = \epsilon(F) = 0$. Define the antipode S such that $S(K) = K^{-1}, S(E) = -E, S(F) = -F$. Thus we have a Hopf algebra structure. Let $q = e^{2i\pi/n}, n$ odd. Then E^n, F^n, K^n are central, and quotienting them yield the finite-dimensional Hopf algebra $u_q(\mathfrak{sl}_2)$. It is quasitriangular with

$$R = \frac{1}{n}(\sum_{a,b=0}^{n-1} q^{-2ab} K^a \otimes K^b) \sum_{r=0}^{n-1} \frac{(q - q^{-1})^r}{[r]_{q^{-2}}!} E^r \otimes F^r, \tag{33}$$

$$[r]_{q^{-21}}! = \prod_{t=1}^{r} [t]_{q^{-2}}, [r]_{q^{-2}} = \frac{1 - q^{-2r}}{1 - q^{-2}} \tag{34}$$

4 Knots

A knot is a smooth simple closed curve in S^3. A link is a disjoint union of a finite number of smooth simple closed curves in S^3. To avoid **wild knots** we only consider knots that are equivalent by ambient isotopy to a smooth curve or a finite union of segments. A major issue is to determine when two knots are equivalent by ambient isotopy.

4.1 Reidemeister Moves

A **Reidemeister move I** undoes a twist. In a braided category, a twist is a map $V \to V^* \to V^{**}$. An r-move I sets $V^{**} = V$, which in a braided category is not the case.

A **Reidemeister move II** undoes an untwisted overlap.

A **Reidemeister move III** is the braid group equivalence.

We want to work with *framed knots*, which are knots with an associated normal vector field. If we flatten the knot out in the direction of the vector field it becomes a ribbon. They can be projected onto the $\mathbb{R}^2$ plane and it can twist. Thus framed knots are related by r-moves II and III but not I. Instead, r-move I yields a double twist.

With Lecture 2 in mind, we can find ev_V when $V = U \otimes W$.

$$ev_{U\otimes W} = ev_W(1_{W^*} \otimes ev_U \otimes 1_W) \quad (35)$$

$$coev_{U\otimes W} = (1_U \otimes coev_W \otimes 1_{U^*})coev_U \quad (36)$$

Lemma 5. $(ev_V \otimes 1_W)(1_{V^*} \otimes c_{W,V}) = (1_W \otimes ev_V)(c^{-1}_{W,V^*} \otimes 1_V)$

Proof.

$$\begin{array}{ccccc} W \otimes V^* \otimes V & \xrightarrow{c_{W,V^*\otimes V}} & V^* \otimes V \otimes W & \xrightarrow{ev_V \otimes 1_W} & K \otimes W \\ & \searrow^{c_{W,V^*}\otimes 1_V} & \uparrow{\scriptstyle 1_{V^*}\otimes c_{W,V}} & & \\ & & V^* \otimes W \otimes V & & \end{array}$$

□

5 (Framed) tangles, naive trace and ribbon trace.

5.1 Category theory of low-dimensional topology

The generalization of braids, knots and links are called **tangles**. A **tangle** is a collection of circles and arcs piecewise smoothly immersed in $\mathbb{R}^2 \times [0,1]$ with endpoints on the planes $\mathbb{R}^2 \times \{0\}, \mathbb{R}^2 \times \{1\}$. Let m, n be nonnegative integers. Fix m points in $\mathbb{R}^2 \times \{0\}$ and fix n points in $\mathbb{R}^2 \times \{1\}$. These are the endpoints of the arcs, and define a tangle of type (m, n). The category of tangles have objects in $\mathbb{N}$. Think of a tangle (m, n) as a morphism $m \to n$. Tangles are drawn with m at the top and projected to $\mathbb{R}^2$. Morphisms are composed by gluing $(k, 0, 1)$ to $k', 0, 1$, $1 \le k \le n, 1 \le k' \le m'$ and rescaling to fit in our 1-interval.

The monoidal structure identifies m_1 and m_2 with $m_1 + m_2$. Given tangles $\mathrm{Hom}(m_1, n_1)$ and $\mathrm{Hom}(m_2, n_2)$, we can compose them to get a tangle in $\mathrm{Hom}(m_1 + m_2, n_1 + n_2)$. We can define $m^* = m$, giving the tangle category a rigid structure. For instance, the coevaluation map for $m = 2$ is given by $\mathrm{Hom}(0, 2 \otimes 2*)$, an object in Hom(0,4). One can introduce a braiding by specifying morphisms in $\mathrm{Hom}(m \otimes n, n \otimes m)$, that is, taking m strands and pulling them over n strands. A **framed tangle** associates to each strand a family of normal vectors. Extend the strand in the direction of these vectors and one has a ribbon. Framed tangles form a braided monoidal category.

One can try to model a knot in terms of a rigid braided category. Assume $V^{**} \cong V$, so $coev_{V^*} : V * \times V^{**} \to k$ gives $V^* \times V \to K$. Interprect caps and cups as evaluation and coevaluation, a knot is a morphism $K \to K$.

The simplest knot is a circle $coev_V \circ ev_{V^*} : K \to V \otimes V^* \to K$. More generally we can take an endomorphism of V and calculate its **trace**: $K \xrightarrow{coev_V} V \otimes V^* \xrightarrow{f \otimes 1_{V^*}} V \otimes V \xrightarrow{ev_{V^*}}$. In the symmetric category of vector spaces, if $f : V \to V, g : W \to W, tr(f \otimes g) = tr(f)tr(g)$

Trying to make a trace for a braided category doesn't work so well. We create $V \otimes V^*$ with $coev_V$. $K \to V \otimes V^* \xrightarrow{c_{V,V^*}} V^* \otimes V \to K$. Diagrammatically, this looks like a figure eight with, in the intersection, the left strand on top (the left strand represents V). We find the trace for a tensor product, the trace is not multiplicative. Draw this out if you don't believe me. This points to the fact that we need something else to make a good theory.

In a rigid braided category, V, V^{**} are naturally isomorphic. There are potentially many isomorphisms between increasingly complicated tangles. For example,

$$u_V : V \xrightarrow{1_V \otimes coev_{V^*}} V \otimes V^* \otimes V^{**} \xrightarrow{c_{V,V^*} \otimes 1_{V^{**}}} V^* \otimes V \otimes V^{**} \xrightarrow{ev_V \otimes 1_{V^{**}}} V^* * \tag{37}$$

is an isomorphism with

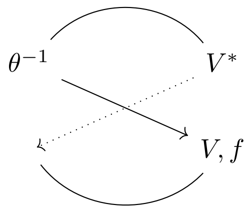
$$u_V^{-1} : V^{**} \xrightarrow{1_{V^{**}} \otimes ev_V} V^{**} \otimes V \otimes V^* \xrightarrow{c_{V^{**},V} \otimes 1_{V^*}} V \otimes V^{**} \otimes V^* \xrightarrow{1_V \otimes ev_{V^*}} V \tag{38}$$

One can easily diagrammatically check that $u_V u_V^{-1} = u_V^{-1} u_V = 1_V$. Consider another isomorphism $V^{**} \to V$ NOT equal to u_V^{-1}:

$$v_V : V^{**} \xrightarrow{1_{V^{**}} \otimes coev_V} V^{**} \otimes V \otimes V^* \xrightarrow{1_{V^{**}} \otimes c_{V^*,V}} V^{**} \otimes V^* \otimes V \xrightarrow{ev_{V^*} \otimes 1_V} V \tag{39}$$

Diagrammatically, v_V, u_V^{-1} correspond to a loop where the crossover in the diagram is over and under, respectively. The difference between v_V, u_V^{-1} is clear if $V = U \otimes W$. Illustrate this to convince yourself. Both u_V, v_V are counter-clockwise twists 2π. Composing them, we get a 4π twist. We can solve many problems such as the non-multiplicativity of the trace if we had a map $V \to V$ that is a 2π twist. With multiple compositions, we can construct morphisms $V \to V$ that are twistings multiples of 4π.

But what we need is a 2π twist. Thus we need a $\theta : V \to V$ such that $\theta^2 = v_V \circ u_V$. Thus we can construct a multiplicative trace:



enter picture here

Be careful, though. $\theta^{-1}_{U\otimes W}$ isn't a twist in $U \otimes W$, it's a twist in U and a twist in W, with the strands then braided together twice.

picture here

A braided category with a twist is called a ribbon category. We want naturality, such that if $f : V \to W, \theta_W f = f\theta_V$, in partiulcar this means, for evals and coevals, we can move θ back and forth. We also must have

$$\theta^{-1}_{U\otimes W} = c_{W,U} \circ c_{U,W} \circ \theta^{-1}_U \otimes \theta^{-1}_W = \theta^{-1}_U \otimes \theta^{-1}_W \circ c_{W,U} \circ c_{U,W} \tag{40}$$

Now we must assume that if I is the unit object of the category, then $\theta_I = 1_I$, $\theta_{V^*} = \theta^*_V$

The definition of a category gives all we need for a multiplicative trace. It's an endomorphism of I.

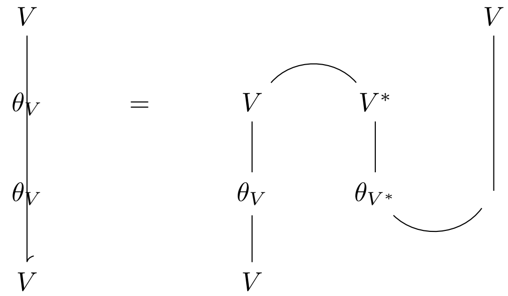
6 Tangle and ribbon categories. Maps and elements of quasitriangular Hopf algebras.

For the tangle category, objects are nonnegative integers, and we can compose their diagrams by composition and by tensor. Given tangles $T \in \text{Hom}(a, b)$ and $U \in \text{Hom}(b, c)$, we define their composition by attaching the b endpoints of T with the b starting points and rescaling to $\mathbb{R}^2 \times [0, 1]$. We define their tensor product as the natural diagram coming from shifting the b endpoints of U to the right of the a endpoints of T. A framed tangle is our first example of a ribbon category. Recall that a framed tangle adds a normal vector field to each curve of the tangle. At the endpoints of the arcs adjoining the boundary, the direction is to be (0,-1,0).

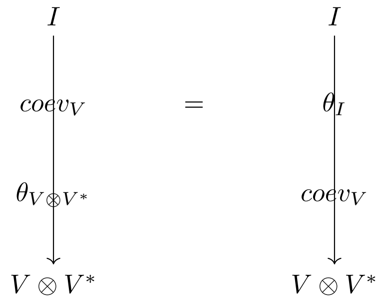
6.1 Ribbon Axioms and θ^2_V

We now prove that $\theta^2 = u_V \circ v_V$, for u_V, v_V as before. By the way, u_V, v_V commute. We begin by taking the ribbon axiom $\theta^{-1}_{U\otimes W} = \theta^{-1}_U \otimes \theta^{-1}_W \circ c_{U,W} \circ c_{W,U}$. This is equivalent to $\theta_U \otimes \theta_W =$

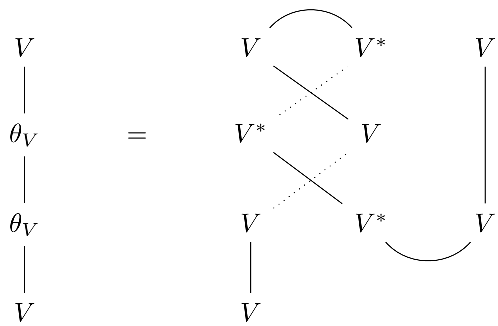
$c_{W,U} \circ c_{U,W} \circ \theta_{U\otimes W}$. By the adjointness of θ and the straightening property of the dual,



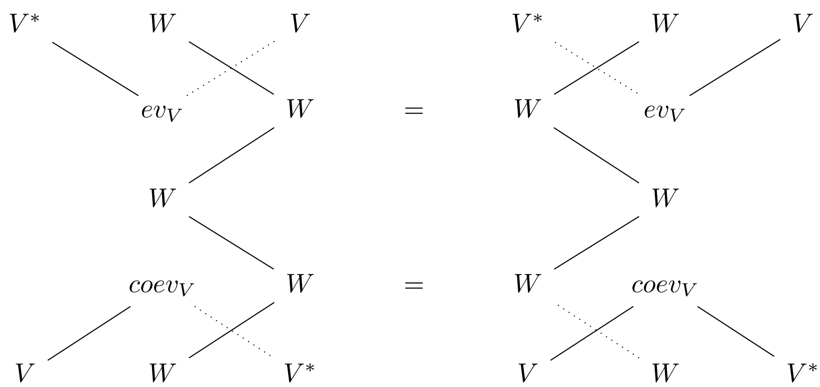
we can discard $\theta_{V\otimes V^*}$ using naturality. Representing the unit object as I, we use the naturality to move the θ map past the *coev* map.



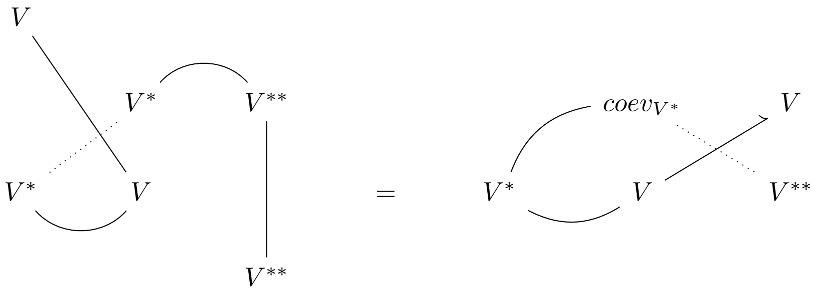
Recall the axiom that $\theta_I = 1_I$. Thus we've proved



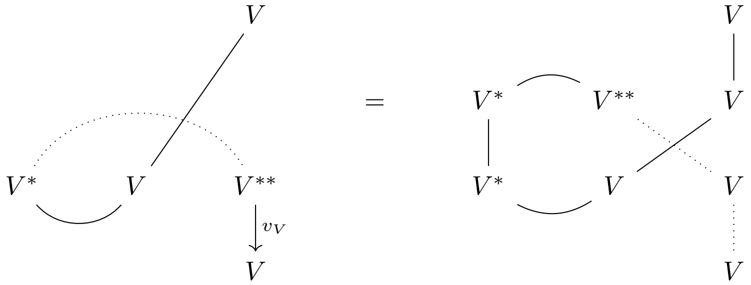
In proving that this is equivalent to $u_V \circ v_V$, use the coevaluation and evaluation crossing identities: $(ev_V \otimes 1_W)(1_{V^*} \otimes c_{W,V}) = (1_W \otimes ev_V)(c^{-1}_{W,V^*} \otimes 1_V)$.



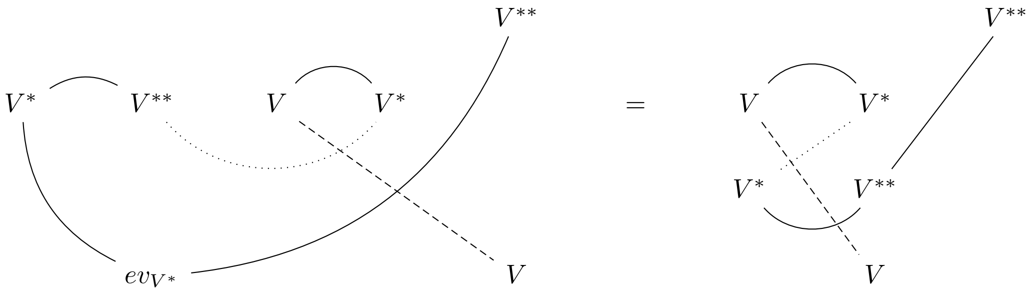
We can rewrite u_V as the following diagram.



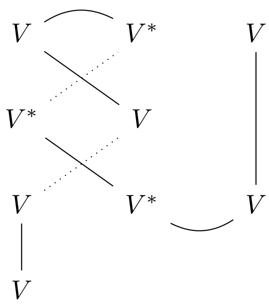
In the same way, we rewrite $v_V u_V$ as the following:



where we used the naturality of the braiding. Now we sub in v_V, and use the straightening property of the dual:



Then use the evaluation crossing identity and we've proved that this is equal to



and thus to θ^2.

6.2 Quasitriangular Hopf Algebras

For a quasitriangular Hopf Algebra, recall that the category of modules over QTHAs are braided monoidal categories. Recall the axiom $\tau\Delta(x) = R\Delta(x)R^{-1}$. For $R = 1$, this means that H is cocommutative. Generally, the existence of R makes H into a *coboundary Hopf algebra.* Writing the braid axioms,

$$(\Delta \otimes 1)R = R_{13}R_{23}, (1 \otimes \Delta)R = R_{13}R_{12} \tag{41}$$

In Sweedler notation, we can write

$$(\Delta\otimes)(R^{(1)})\otimes R^{(2)} = R^{(1)}_{(1)}\otimes R^{(1)}_{(2)}\otimes R^{(1)} \tag{42}$$

In keeping track of the second copy of R, we add a tilde to it:

$$R^{(1)}_{(1)}\otimes R^{(1)}_{(2)}\otimes R^{(2)} = R^{(1)}\otimes \tilde{R}^{(1)}\otimes R^{(2)}\tilde{R}^{(2)}, \tag{43}$$

$$R^{(1)}\otimes R^{(2)}_{(1)}\otimes R^{(2)}_{(2)} = R^{(1)}\tilde{R}^{(1)}\otimes \tilde{R}^{(2)}\otimes R^{(2)} \tag{44}$$

Recall that scalars can be moved around:

$$x_{(1)}\epsilon(x_{(2)}) = \epsilon(x_{(1)})x_{(2)} = x \tag{45}$$

Lemma 6. $(1\otimes\epsilon)R = (\epsilon\otimes 1)R = 1_{H\otimes H}$

Proof. In Sweedler notation, this is $R^{(1)}\otimes\epsilon(R^{(2)})$ which, since ϵ is scalar-valued, is equal to $R^{(1)}\epsilon(R^{(2)})\otimes 1_H$. We want to show, then, that $R^{(1)}\epsilon(R^{(2)}) = 1_H$. We have that $(\epsilon\otimes 1)\Delta(x) = 1\otimes x$. Then $(\epsilon\otimes 1\otimes 1)(\Delta\otimes 1)R = 1\otimes R^{(1)}\otimes R^{(2)} = R_{23}$. Plug in the braid identity $(\Delta\otimes 1)R = R_{13}R_{23}$. Since R_{23} is invertible, we have, since $(\epsilon\otimes 1\otimes 1)(\Delta\otimes 1)R = (\epsilon\otimes 1\otimes 1)R_{13}R_{23}$, $(\epsilon\otimes 1\otimes 1)R_{13} = 1_{H\otimes H\otimes H}$. Thus $(\epsilon\otimes 1)R = 1$, and the case $(1\otimes\epsilon)R = 1$ is similar. □

Theorem 2. $R_{12}R_{23}R_{13} = R_{23}R_{13}R_{12}$

Proof. Apply the axiom $\tau\Delta(x) = R\Delta(x)R^{-1}$. Use the fact that $R_{23}R_{13} = (\tau\otimes 1)R_{13}R_{23} = R_{12}R_{23}R_{13}R_{12}^{-1}$ We get $(\tau\otimes 1)(\Delta\otimes 1)R = R_{12}((\Delta\otimes 1)R)R_{12}^{-1}$. Rearranging yields the Yang-Baxter equation we were trying to prove. □

Now we prove the identities

$$(S\otimes 1)R = R^{-1}, (1\otimes S)R^{-1} = R \tag{46}$$

These imply $(S\otimes S)R = R$

7 The standard module of $U_q(\mathfrak{sl}_2)$ and its R-matrix, the 6-

The following representation theories are all the same:

Finite-dimensional representations of $SL(2,\mathbb{R})$

Finite-dimensional representations of $SU(2)$

Finite-dimensional analytic representations of $SL(2,\mathbb{C})$

Finite dimensional complex representations of their Lie algebras: $\mathfrak{sl}_2(\mathbb{R}), \mathfrak{su}_2, \mathfrak{sl}_2(\mathbb{C})$

The enveloping algebras of these Lie algebras

The universal quantized enveloping algebra $U_q(\mathfrak{sl}_2)$,

where the categories have the same simple objects, but $U_q(\mathfrak{sl}_2)$ has an interesting comultiplication and R-matrix, so they are not the same in terms of braided categories. Each of the above categories are semisimple, and there is one unique irreducible representation in every dimension. The two-dimensional standard representation V generates the category, i.e. every irreducible is a submodule of $\otimes^k V$ for some k: the k-dimensional irreducible submodule is the symmetric power $\mathrm{Sym}^{k-1}(V)$. The standard module of $\mathfrak{sl}_2$ has two basis vectors X, Y. With respect to this basis,

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $H = [E, F]$. For the 4D irreducible, we have

$$E = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

The standard module of $U_q(\mathfrak{sl}_2)$ is given similarly:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

$$E = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & [2] & 0 \\ 0 & 0 & 0 & [3] \\ 0 & 0 & 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ [3] & 0 & 0 & 0 \\ 0 & [2] & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, H = \begin{pmatrix} q^3 & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & q^{-3} \end{pmatrix}$$

for $[n] = \frac{q^n - a^{-n}}{q - q^{-1}}$, the 'quantum integer.'