Alec Lau's Favorite Algorithms

1 Grover's Algorithm

Given a set of N elements $X = \{x_1, ..., x_N\}$ along with a boolean function $f : X \to \{0, 1\}$, we wish to find some $x_i \in X$ such that $f(x_i) = 1$.

Definition 1. *The 2^n-dimensional Hadamard gate is defined recursively as*

$$H_1 = \begin{pmatrix} 1 \end{pmatrix}, H_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, H_{2^n} = \frac{1}{\sqrt{2^n}} \begin{pmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{pmatrix} \tag{1}$$

One can verify that, given a string of qubits $|0\rangle^{\otimes n}$, H_{2^n} puts the qubits in an equal weight superposition.

Definition 2. *The Z_0 operator is given by $2|0\rangle^{\otimes n}\langle 0|^{\otimes n} - I$.*

One can verify that $Z_0|x\rangle$ is equal to $|x\rangle$ if $|x\rangle = |0\rangle^{\otimes n}$, and $-|x\rangle$ if not (orthogonality is a beautiful thing).

Definition 3. *The Grover Diffusion Gate is given by the following gate: $D := H^{\otimes n} Z_0 H^{\otimes n}$.*

It is a straightforward computation to prove that $D|x\rangle = \sum_{x \in \{0,1\}^n} (2\mu - \alpha_x)|x\rangle$, where α_x is the amplitude of $x \in \{0, 1\}^n$, and $\mu = \frac{1}{N}\sum_x \alpha_x$

Definition 4. *The $O_f^{\pm}$ gate is given by the operator $(-1)^{f(x)}$ operating on $|x\rangle$.*

The Algorithm

$|0\rangle^{\otimes n} \to H_{2^n} \to O_f^{\pm} D \to ...[\mathcal{O}(\sqrt{N})\text{times}]... \to O_f^{\pm} D \to$ measurement.

The Hadamard matrix induces an equal weight superposition in our string of 0s. Next, the $O_f^{\pm}$ gate isolates the wanted bit by flipping the sign of its amplitude. D takes all amplitudes and reflects them about the average, which is positive. Since the amplitude of our wanted bit is negative, the amplitude after D gets flipped to a larger number than the others. Keep applying this gate, and the amplitude of the wanted bit becomes larger and larger so that, when we finally measure, we are almost certainly going to observe the desired bit! :o

Analysis

Regular unsorted search: $\mathcal{O}(n)$. Quantum search via Grover's Algorithm: $\mathcal{O}(\sqrt{N})$.

2 (Racah-Speiser) or (Brauer-Klimyk) Algorithm

We wish to decompose tensor products of irreducible representations of a Lie group. Given dominant integral weights λ, μ, compute $N^{\nu}_{\lambda\mu}$.

Definition 5. *For W the Weyl group of Lie algebra , the shifted reflection of a weight φ, for some $\omega \in W$ is given by*

$$\omega \cdot \varphi = \omega(\varphi + \rho) - \rho \tag{2}$$

Definition 6. *Let $m_\lambda(\mu)$ denote the weight space V_μ in the irreducible representation of $V(\lambda)$.*

Derivation

(Using Einstein's notation where clear)

$$\chi_{V_\lambda \otimes V_\mu} = \chi(V_\lambda) \cdot \chi(V_\mu) = N^{\nu}_{\mu\lambda}\chi_\nu \tag{3}$$

$$\sum_{\omega\in\mathcal{W}} sign(\omega) e^{\omega(\lambda+\rho)} \cdot \chi(V_\mu) = \sum_{\omega \in W} sign(\omega) N^{\nu}_{\mu\lambda} e^{\omega(\nu+\rho)} \tag{4}$$

$$\sum_{\omega\in\mathcal{W}} sign(\omega) \sum_{\kappa} mult_\mu(\kappa) e^{\omega(\lambda+\rho)} e^{\kappa} = \sum_{\omega\in\mathcal{W}} sign(\omega) N^{\nu}_{\mu\lambda} e^{\omega(\nu+\rho)} \tag{5}$$

$$\sum_{\omega\in\mathcal{W}} sign(\omega) \sum_{\kappa\in V_\mu} mult_\mu(\kappa) e^{\omega(\lambda+\rho+\kappa)} = \sum_{\omega\in\mathcal{W}} sign(\omega) N^{\nu}_{\mu\lambda} e^{\omega(\nu+\rho)} \tag{6}$$

$$\tag{7}$$

Now we have a set of linear combinations of exponentials e^η, which are linearly independent for each η. Since the Weyl group acts on η by sending it through the different Weyl chambers, there is one ω such that $\omega(\eta)$ is in the fundamental Weyl chamber. Thus we need only consider e^η with η in the Weyl chamber.

Let $\eta = \lambda + \kappa + \rho$. Assume that η is in the interior of the fundamental Weyl chamber. Then only the identity of the Weyl group keeps our exponentials with weights in the fundamental Weyl chamber, simplifying our expression to

$$mult_\mu(\kappa) e^{\omega(\eta)} = N^{\nu}_{\lambda\mu} e^{\omega(\nu+\rho)} \Rightarrow \tag{8}$$

$$mult_\mu(\nu - \lambda) = N^{\nu}_{\lambda\mu} \Rightarrow \tag{9}$$

$$mult_\mu(\nu) = N^{\lambda+\nu}_{\lambda\mu} \tag{10}$$

for $\eta \in P^o_+, \forall \nu \in V_\mu$

What if $\eta = \lambda + \kappa + \rho, \kappa \in V_\mu$ is not in the fundamental Weyl chamber? Pick a $\kappa \in V_\mu$. We know there must exist some $\omega \in W$ such that $\omega(\eta) \in P_+^o$. No fear! Then we have

$$sign(\omega)e^{\omega(\eta)}mult_\mu(\kappa) \tag{11}$$

Sum this up, and you have that it must match some $N_{\lambda\mu}^{\nu}e^{\nu+\rho}$ via (4). Thus we have

$$\omega(\eta) = \nu + \rho \Rightarrow \tag{12}$$

$$\kappa = \omega^{-1}(\nu + \rho) - \rho - \lambda \Rightarrow \tag{13}$$

$$N_{\lambda\mu}^{\nu} = \sum_{\omega \in W} sign(\omega)mult_\mu(\omega^{-1}(\nu + \rho) - \rho - \lambda) \tag{14}$$

The Algorithm

Set $N_{\lambda\mu}^{\nu} = 0$.

(a) Construct the weight diagram of λ, denoted here are $WD(\lambda)$.

(b) Translate each weight in $WD(\lambda)$ by μ.

(c) For each weight $\varphi \in WD(\lambda) + \mu$, check to see if φ is not fixed by any shifted reflection $\omega \cdot \varphi, \omega \in W$.

(d) If not, compute a $\omega^* \in W$ such that $\omega^* \cdot \varphi \in P_+^o$

Add $sign(\omega^*)m_\lambda(\varphi - \mu)$ to $N_{\lambda\mu}^{\omega^*\cdot\varphi}$.

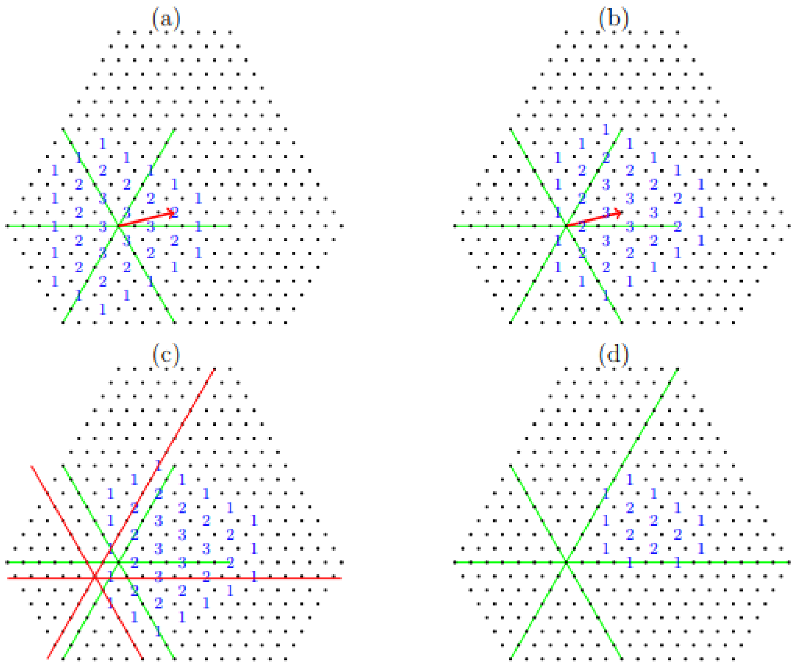

Image from Barker, et. al. "A new proof of a formula for the type A_2 fusion rules"

Remark 1. *One can get the **Kac-Walton Algorithm** by replacing the Weyl group with the affine Weyl group, and replacing the fundamental Weyl chamber with the fundamental Weyl alcove, given by* $P_l := \{\beta \in$

$P_+^o : (\beta,\theta) \leq l\}$, *where* θ *is the highest root, and* l *is a fixed integer called the 'level' of the fundamental alcove. This algorithm is used to calculate the fusion coefficients in conformal field theory.*