Algebraic Topology Refresher Problems

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Note: These are the only problems I had saved before I decided to make a page to preserve this stuff. Because there is so much machinery in algebraic topology, I have saved some of these problems as a refresher for whenever I return to it, as said machinery is very forgettable, beautiful though it may be. Also, these problems were done open-book and open-past-homework with Hatcher's "Algebraic Topology" textbook, so there may be some steps I skipped over.

Denotation: $\mathbb{R}P^n$ is the space of all lines through the origin in $\mathbb{R}^n$, S^n is the $n-$dimensional sphere, and D^n is the $n-$dimensional disc.

All problems were written by Professor Ralph Cohen, saxophone extraordinaire.

Question 1. *Is every covering space of $\mathbb{R}P^2 \times \mathbb{R}P^3$ isomorphic to a product of covering spaces $p_1 \times p_2 : \tilde{X}_1 \times \tilde{X}_2 \to \mathbb{R}P^2 \times \mathbb{R}P^3$, where $p_1 : \tilde{X}_1 \to \mathbb{R}P^2$ and $p_2 : \tilde{X}_2 \to \mathbb{R}P^3$? Why or why not?*

Proof. Let Y be the covering space of $\mathbb{R}P^2 \times \mathbb{R}P^3$. If there exists an isomorphism $f : Y \to \tilde{X}_1 \times \tilde{X}_2$, then from the relations $p_1 = p_2 f, p_2 = p_1 f$, it follows that $(p_1 \times p_2)_* \pi_1(\tilde{X}_1 \times \tilde{X}_2) \cong p_*(Y)$ due to the induced isomorphisms. Since $\mathbb{R}P^2$ and $\mathbb{R}P^3$ are path-connected, we know from the product topology that a map $f : Y \to \mathbb{R}P^2 \times \mathbb{R}P^3$ is continuous if and only if the maps $g : Y \to \mathbb{R}P^2, h : Y \to \mathbb{R}P^2$ defined by $f = g \times h$ are continuous. Therefore a loop in $\mathbb{R}P^2 \times \mathbb{R}P^3$ is equivalent to a pair of loops in $\mathbb{R}P^2$ and $\mathbb{R}P^3$, and furthermore a homotopy on the product space is equivalent to a pair of homotopies on the corresponding components. Thus there exists a bijection

$$\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^3) \cong \pi_1(\mathbb{R}P^2) \times \pi_1(\mathbb{R}P^3) \tag{1}$$

given by

$$[f] \mapsto ([g], [h]) \tag{2}$$

Next we compute the homology groups of $\mathbb{R}P^n, n \in \{2, 3\}$.

$\mathbb{R}P^n$ is topologized as the quotient space $\mathbb{R}^{n+1} - \{0\}$ under the equivalence relation $v \sim \lambda v$ for scalars $\lambda \neq 0$, so we can thus restrict to vectors of length 1, so $\mathbb{R}P^n = S^n/(v \sim -v)$. Thus $\mathbb{R}P^n$ is the quotient space of a hemisphere D^n with antipodal points of ∂D^n identified. Since ∂D^n with antipodal points identified is just $\mathbb{R}P^{n-1}$, we see that $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an $n-$cell, with the quotient projection $S^{n-1} \to \mathbb{R}P^{n-1}$ as the attaching map. By induction on n, $\mathbb{R}P^n$ has a CW structure $e^0 \cup e^1 \cup ... \cup e^n$ with one cell e^i in each dimension $i \leq n$. To compute the boundary map d_k we compute the degree of the composition

$$S^{k-1} \xrightarrow{\varphi} \mathbb{R}P^{k-1} \xrightarrow{q} \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} = S^{k-1} \tag{3}$$

with q the quotient map. The map $q\varphi$ is a homeomorphism when restricted to each component of $S^{k-1} - S^{k-2}$, and these two homeomorphisms are obtained from each other by precomposing with the antipodal map of S^{k-1}, which has degree $(-1)^k$. Hence deg$q\varphi =$ deg(1) $+$ deg(-1) $= 1 + (-1)^k$, and so the boundary maps d_k is either 0 or multiplication by 2, depending on whether k is odd or even. Thus the cellular chain complex for $\mathbb{R}P^n$ is

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} ... \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \text{ if } n \text{ is even} \tag{4}$$

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} ... \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \text{ if } n \text{ is odd} \tag{5}$$

$$\tag{6}$$

It follows that

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} \text{ if } k = 0 \text{ and for } k = n \text{ odd} \\ \mathbb{Z}_2 \text{ if } k = \text{ odd, } 0 < k < n \\ 0 \text{ otherwise} \end{cases} \tag{7}$$

Thus we know that $H_1(\mathbb{R}P^2) \cong H_1(\mathbb{R}P^3) \cong \mathbb{Z}_2$. We know from homework 4 that H_1 is the abelianization of π_1 (since $\mathbb{R}P^n$ is path-connected and nonempty), but a group with cardinality 2 must be isomorphic to $\mathbb{Z}_2$ to have the group axioms still hold. Thus, $\pi_1(\mathbb{R}P^2) \cong \pi_1(\mathbb{R}P^3) \cong \mathbb{Z}_2$, and $\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^3) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. $\mathbb{R}P^2$ and $\mathbb{R}P^3$ are path-connected, and are manifolds, and so are locally path-connected semi-locally simply-connected as well. Thus we can apply the Galois Correspondence Theorem to say that, for every subgroup H of $\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^3) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, there is an isomorphism class of covering spaces Y such that $p_*(Y) \cong H$, therefore all covering spaces of $\mathbb{R}P^2 \times \mathbb{R}P^3$ have their fundamental groups as subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

The subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are the trivial subgroup (0,0), and groups with generator (1,0), (0,1), and (1,1). I claim that there is no pair of maps $p_1 : \tilde{X}_1 \to \mathbb{R}P^2, p_2 : \tilde{X}_2 \to \mathbb{R}P^3$ such that $(p_1 \times p_2)_*(\tilde{X}_2 \times \tilde{X}_3) \cong \{(0,0),(1,1)\}$. We see that $|\{(0,0),(1,1)\}| = 2$. By Lagrange's Theorem for any subgroup of this group must have cardinality that divides 2. Since 2 is prime, the only subgroup in it has cardinality 1 and is thus is the trivial subgroup. Thus, if we have $\pi_1(\tilde{X}_1) \times \pi_1(\tilde{X}_2) \cong \{(0,0),(1,1)\}$, a subgroup $\pi_1(\tilde{X}_1) \times 0 \cong 0$ and another subgroup $0 \times \pi_1(\tilde{X}_2) \cong 0$. Thus both $\pi_1(\tilde{X}_1), \pi_1(\tilde{X}_2)$ are trivial subgroups, and it is impossible for $f((0,0)) \mapsto (1,1)$ if f is an isomorphism. If, without loss of generality $\pi_1(\tilde{X}_1) \times 0$ were the whole group, this has cardinality 2, but the group is $\{(0,0),(1,0)\}$, a different subgroup corresponding to a different covering space. Thus, $\{(0,0),(1,1)\} \not\cong \pi_1(\tilde{X}_1) \times \pi_1(\tilde{X}_2)$, so not all covering spaces of $\mathbb{R}P^2 \times \mathbb{R}P^3$ are isomorphic to a product of each summand's covering space. □

Question 2. *Let $X = \mathbb{R}P^2 \vee S^3$ and $Y = \mathbb{R}P^3$. Prove that the homology and cohomology groups of X and Y are isomorphic with any coefficients, but that X and Y do not have the same homotopy type.*

Proof. From the above, we know that the chain complex for $\mathbb{R}P^3$ is

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \tag{8}$$

With any G coefficients, this becomes

$$0 \to G \xrightarrow{0} G \xrightarrow{2} G \xrightarrow{0} G \to 0 \tag{9}$$

so we have

$$H_0(\mathbb{R}P^3; G) \cong G, H_1(\mathbb{R}P^3; G) \cong G/2G, H_2(\mathbb{R}P^3; G) \cong 0, H_3(\mathbb{R}P^3; G) \cong G, \tag{10}$$

$$H_0(\mathbb{R}P^2; G) \cong G, H_1(\mathbb{R}P^2; G) \cong G/2G, H_2(\mathbb{R}P^2; G) \cong 0 \tag{11}$$

For $n > 0$ take $(X, A) = (D^n, S^{n-1})$ so $X/A = S^n$. The terms $\tilde{H}_i(D^n)$ in the long exact sequence for this pair are zero since D^n is contractible. Exactness of the sequence then implies that the maps $\tilde{H}_i(S^n) \xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1})$ are isomorphisms for $i > 0$ and that $\tilde{H}_0(S^n) = 0$ By induction on n, starting with the case of S^0, we see that $\tilde{H}_n(S^n) \cong \mathbb{Z}$ and $\tilde{H}_i(S^n) = 0$ for $i \neq n$. Thus $H_1(S^3; G) \cong H_2(S^3; G) \cong 0, H_3(S^3) \cong G)$, due to the equivalence of $\tilde{H}_n$ and H_n for $n > 0$. Since $S^3, \mathbb{R}P^2$ are both path-connected and nonempty, $S^3 \vee \mathbb{R}P^2$ is path-connected and nonempty. By definition, $H_0(S^3 \vee \mathbb{R}P^2) = C_0(S^3 \vee \mathbb{R}P^2)/\text{Im } \partial_1$ since $\partial_0 = 0$. Define a homomorphism $\epsilon : C_0(S^3 \vee \mathbb{R}P^2) \to \mathbb{Z}$ by $\epsilon(\sum_i n_i \sigma_i) = \sum_i n_i$. This is obviously surjective since $S^3 \vee \mathbb{R}P^2$ is nonempty. Ker $\epsilon = \text{Im } \partial_1$ since $S^3 \vee \mathbb{R}P^2$ is path-connected, and thus ϵ induces an isomorphism.

We conclude that $H_0(S^3 \vee \mathbb{R}P^2; G) \cong G$. Since reduced homology is the same as homology relative to a basepoint, we know that, for $n > 0$,

$$\tilde{H}_n(S^3 \vee \mathbb{R}P^2) \cong H_n(S^3 \vee \mathbb{R}P^2) \cong H_n(S^3) \oplus H_n(\mathbb{R}P^2) \tag{12}$$

Thus we have $H_1(S^3 \vee \mathbb{R}P^2; G) \cong G/2G, H_2(S^3 \vee \mathbb{R}P^2; G) \cong 0, H_3(S^3 \vee \mathbb{R}P^2; G) \cong G$. These are the same (up to isomorphism) homology groups as $\mathbb{R}P^3$.

In calculating cohomology for any G coefficients, we notice that $H^n(X; G) \cong \text{Hom}(H_n(X; \mathbb{Z}), G) \oplus \text{Ext}(H_{n-1}(X; \mathbb{Z}), G)$.

Lemma 1. *Hom*$(\mathbb{Z}, G) \cong G$*, Hom*$(G/2G, G) \cong 0$

Proof. By mapping 1 to each element of G, we get a cardinality of G. Since this is a homomorphism, the structure of the image is preserved, and $fg(n) = f(n) \star g(n)$, where $\star$ is the group operation of G. Since every element of the group is hit in the image, and the composition of these homomorphisms is mapped to the group operation, we have all elements of G following the same structure of G, and thus is isomorphic to G.

In order for there to be a nontrivial homomorphism, orders of elements must match from $G/2G$ to G. However, this is not the case, as we mod out by $2G$, so no generator of $G/2G$ has the same order as an element in G. Thus, the only homomorphism possible is the trivial homomorphism. □

Using this, we calculate cohomology for $\mathbb{R}P^3$, and, using the rules of Ext on page 195 of Hatcher, we find

$$H^0(\mathbb{R}P^3;G) \cong G, H^1(\mathbb{R}P^3;G) \cong 0 \oplus \mathrm{Ext}(\mathbb{Z},G) \cong 0 \oplus 0 \cong 0, \tag{13}$$

$$H^2(\mathbb{R}P^3;G) \cong 0 \oplus \mathrm{Ext}(\mathbb{Z}_2,G) \cong 0 \oplus G/2G \cong G/2G, \tag{14}$$

$$H^3(\mathbb{R}P^3;G) \cong G \oplus 0 \tag{15}$$

We also have

$$H^n(S^3 \vee \mathbb{R}P^2;G) \cong \mathrm{Hom}(H_n(S^3 \vee \mathbb{R}P^2;\mathbb{Z}),G) \oplus \mathrm{Ext}(H_{n-1}(S^3 \vee \mathbb{R}P^2;\mathbb{Z}),G) \tag{16}$$

Since the homology groups are isomorphic, the cohomology groups are isomorphic as well.

If we can prove $H^*(\mathbb{R}P^2 \vee S^3;\mathbb{Z}_2) \ncong H^*(\mathbb{R}P^3;\mathbb{Z}_2)$, then this means the spaces are not homotopy equivalent. Plug in $G = \mathbb{Z}_2$. From Example 3.8 in Hatcher, we have $H^*(\mathbb{R}P^2;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^3)$, and $H^*(\mathbb{R}P^3;\mathbb{Z}_2) \cong \mathbb{Z}_2[\beta]/(\beta^4)$, where $|\alpha| = |\beta| = 1$. Suppose we have $\beta \in H^1(\mathbb{R}P^3;\mathbb{Z}_2)$. Then $\beta \smile \beta = \beta^2 \in H^2(\mathbb{R}P^3;\mathbb{Z}_2)$, and $\beta^2 \smile \beta = \beta^3 \in H^3(\mathbb{R}P^3;\mathbb{Z}_2) \cong \mathbb{Z}_2$, and $\beta^3 \neq 0$. Noticing that $H^1(\mathbb{R}P^2 \vee S^3) \cong H^1(\mathbb{R}P^2) \oplus H^1(S^3)$, we have $(\alpha, 0) \in H^1(\mathbb{R}P^2 \vee S^3)$. Suppose there is an isomorphism $f : \mathbb{R}P^2 \vee S^3 \to \mathbb{R}P^3$. Then for $\alpha' = f(\beta) \in H^1(\mathbb{R}P^2;\mathbb{Z}_2)$, and for $a = f(\beta^2) \in H^2(\mathbb{R}P^2;\mathbb{Z}_2)$. We now have $(\alpha', 0) \smile (a, 0) = (\alpha' a, 0) \in H^3(\mathbb{R}P^2 \vee S^3)$. But $H^3(\mathbb{R}P^2;\mathbb{Z}_2) \cong 0$, so $\alpha' a = 0$. But since f is a ring isomorphism, then $f(\beta^3 \neq 0) = \alpha' a = 0$. Since f maps a nonzero element to 0, it cannot be an isomorphism, so the cup product structures of $\mathbb{R}P^2 \vee S^3$ and $\mathbb{R}P^3$ are not homotopically equivalent. □

Question 3. *Let M^n be a closed, path connected, orientable manifold. Let $x \in U \subset M$ where U is an open neighborhood homeomorphic to $\mathbb{R}^n$. Consider the "pinch map," $p : M^n \to S^n$ defined as the composition*

$$p : M^n \xrightarrow{quotient} M^n/(M^n - U) \xrightarrow{homeo} S^n \tag{17}$$

Show that

$$p_* : H_n(M^n; \mathbb{Z}) \to H_n(S^n; \mathbb{Z}) \tag{18}$$

is an isomorphism.

Proof. Since M is closed and path-connected, we can apply Theorem 3.26 in Hatcher to conclude that $H_n(M) \cong H_n(M, M - x)$. Hatcher states that the isomorphism between the two factors though $H_n(M, M - U)$ for U any neighborhood in M containing x. This is because the homomorphism $i_* : H_n(M, M - U) \to H_n(M, M - x)$ induced by inclusion is bijective, since $X - U$ is a deformation retract of $X - x$. By excision, $H_n(X, X - U) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - U) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - x)$. Since $\mathbb{R}^n$ is contractible, this is isomorphic to $H_{n-1}(\mathbb{R}^n - U)$. The second map is isomorphic for any $x \in U$, because $\mathbb{R}^n - U$ and $\mathbb{R}^n - x$ deformation retract onto a sphere centered at x. Thus $H_n(M) \cong H_n(M, M - U)$.

We notice that $(M, M - U)$ is a good pair; Simply take an open cover ϵ-thick covering the boundary of U, and add this to $M - U$. Because $U \cong \mathbb{R}^n$, the overlap of our covering with U can easily be deformation retracted until we are left with $M - U$. Since $(M, M - U)$ is a good pair, we can have a neighborhood V be a neighborhood of U in M that deformation retracts onto U. We have a commutative diagram

$$\begin{array}{ccccc}
H_n(M, U) & \longrightarrow & H_n(M, V) & \longleftarrow & H_n(M - U, V - U) \\
\big\downarrow q_* & & \big\downarrow q_* & & \big\downarrow q_* \\
H_n(M/U, U/U) & \longrightarrow & H_n(M/U, V, U) & \longleftarrow & H_n(M/U - U/U, V/U - U/U)
\end{array}$$

The upper left horizontal map is an isomorphism since in the long exact sequence of the triple (M, V, U) the groups $H_n(V, U)$ are zero for all n, because a deformation retraction of V onto U gives a homotopy equivalence of pairs $(V, U) \cong (U, U)$, and $H_n(U, U) = 0$. The deformation retraction of V onto U induces a deformation retraction of V/U onto U/U so the same argument shows that the lower left horizontal map is an isomorphism as well. The other two horizontal maps are isomorphisms directly from excision. The right-hand vertical map q_* is an isomorphism since q restricts to a homeomorphism on the complement of U. From the commutativity of the diagram it follows that the left q_* map is an isomorphism. We see then that $H_n(M, M - U) \cong$

$H_n(M/(M-U),(M-U)/(M-U)) \cong H_n(M/(M-U))$. Thus $H_n(M) \cong H_n(M/(M-U))$. Since $M/(M-U)$ is homeomorphic to S^n, they better have isomorphic homology groups. Call the isomorphism induced by the homeomorphism between $H_n(M/(M-U))$ and $H_n(S^n)$ g, and the isomorphism between $H_n(M)$ and $H_n(M/(M-U))$, shown via the diagram. Now we have that $p_* := f \circ g$, a composition of isomorphisms, so $p_* : H_n(M) \to H_n(S^n)$ is an isomorphism. □

Question 4. *Prove that if M^3 is a closed, simply connected manifold, then there is a map $g : M^3 \to S^3$ that induces an isomorphism in homology groups in all dimensions. This is a weaker statement of the Poincaré Conjecture, proved in 2003 by G. Perelman.*

Proof. Define $\tilde{M} = \{\mu_x | x \in M$ and μ_x is a local orientation of M at $x\}$. The map $\mu_x \mapsto x$ defines a two-to-one surjection, and, due to everything being nice and manifold-y, we see that $\tilde{M}$ is a two-sheeted covering space of M^3.

Since M^3 is simply connected, $\tilde{M}$ has either one or two components since it is a two-sheeted covering space of M^3. If it has two components, they are each mapped nomeomorphically to M^3 by the covering projection, so M^3 is orientable, being homreomorphic to a component of the orientable manifold $\tilde{M}$. Thus M^3 is orientable, and $H_0(M^3) \cong \mathbb{Z}$ because simply connected implies nonempty and path-connected. Now since M^3 is a closed and orientable manifold, we can use Poincaré duality. Also from Theorem 3.26 (part c), $H_i(M^3) \cong 0, i > 3$. Because M^3 is simply-connected, $\pi_1(M^3) \cong 0 \cong H_1(M^3)$. $H^1(M^3) \cong \text{Hom}(0, \mathbb{Z}) \oplus \text{Ext}(\mathbb{Z}, \mathbb{Z}) \cong 0$. By Poincare duality, $H_2(M^3) \cong H^1(M^3) \cong 0$. Also by Poincaré Duality, $H_3(M^3) \cong H^0(M^3) \cong \text{Hom}(H_0(M^3), \mathbb{Z}) \oplus \text{Ext}(0, \mathbb{Z}) \cong \mathbb{Z} \oplus 0 \cong \mathbb{Z}$. To sum this up, we have $H_0(M^3) \cong H^3(M^3) \cong \mathbb{Z}, H_1(M^3) \cong H_2(M^3) \cong 0 \cong H_i(M^3), i > 3$. These are the exact homology groups of S^3, so let g be the isomorphism between their homology groups. □

Question 5. *Is $(S^2 \times S^4) \vee S^8$ homotopy equivalent to a compact closed manifold? Explain.*

Proof. Let $a_i \in H^i(S^2; \mathbb{Z}), b_i \in H^i(S^4; \mathbb{Z})$ be generators of their cohomology groups. From the definition of the external cup product we have $p_1^*(a) \smile p_2^*(b) \in H^*(X \times Y; R)$, for p_1, p_2 projection maps. For $H^0(S^2 \times S^4) \cong \mathbb{Z}$ because this is space and path-connected. Let p_1^*, p_2^* be induced homomorphisms from the projection $S^2 \times S^4 \to S^2, S^2 \times S^4 \to S^4$, respectively. We have $p_1^*(a_1) \smile p_2^*(b_1) = 0 \smile 0 = p_1^*(a_1) \smile 0 = 0 \smile p_2^*(b_1) = 0 \in H^1(S^2 \times S^4; \mathbb{Z})$. We also have $p_1^*(a_2) \smile$

$p_2^*(b_2) = p_1^*(a_2) \smile p_2^*(0) \in H^2(S^2 \times S^4; \mathbb{Z})$ as a generator for H^2. The other nonzero generator b_4, when cupped with another generator $a_i, i \neq 2$ is $0 \smile p_2^*(b_4) \in H^4(S^2 \times S^4; \mathbb{Z})$, which is the generator of H^4. For all other combinations when $a_i \neq a_2, b_j \neq 4$ we have trivial H^{i+j}. With $0 \neq p_1^*(a_2) \smile p_2^*(b_4) \in H^6(S^2 \times S^4; \mathbb{Z})$, we conclude that $H^i(S^2 \times S^4; \mathbb{Z}) \cong \mathbb{Z}$ (has a single generator infinite with $\mathbb{Z}$ coefficients) when $i = 0, 2, 4, 6$ and trivial otherwise. We might think we would run into trouble with $p_1^*(a_2) \smile p_1^*(a_2)$, but because this is the pullback of the generator $H^2(S^2)$ under p_1, and in $H^2(S^2), a_2 \smile a_2 = 0$, this still holds in $H^2(S^2 \times S^4)$. By the Kunneth Formula, we have $H^*(S^2 \times S^4; R) \cong \mathbb{Z}[a_2]/(a_2^2) \otimes_R \mathbb{Z}[a_4]/(a_4^2), |a_2| = 2, |a_4| = 4$.

For $\tilde{H}^*((S^2 \times S^4) \vee S^8)$ (we need not worry about H^0 since the space is nonempty and path-connected), we use the fact from Hatcher that $\tilde{H}^*((S^2 \times S^4) \vee S^8) \cong \tilde{H}^*(S^2 \times S^4) \oplus \tilde{H}^*(S^8)$. For $\tilde{H}^*(S^8)$, we know that $H_i \cong H^i \cong \mathbb{Z}$ for $i = 0, 8$, and $\cong 0$ if else. Thus our cohomology ring is $\tilde{H}^*(S^8; \mathbb{Z}) \cong \mathbb{Z}[b]/(b^2), |b| = 8$. Thus we have $\tilde{H}^*((S^2 \times S^4) \vee S^8) \cong [\mathbb{Z}[a_2]/(a_2^2) \otimes \mathbb{Z}[a_4]/(a_4^2)] \oplus \mathbb{Z}[b]/(b^2), |a_2| = 2, |a_4| = 4, |b| = 8$.

Any manifold homotopically equivalent to $(S^2 \times S^4) \vee S^8$ must be an 8-manifold. From Theorem 3.26 in Hatcher, if a manifold is not oriented, then $H_8(M; \mathbb{Z}) \cong 0 \Rightarrow H^8(M; \mathbb{Z}) \cong 0$, which cannot be possible, as $H^8((S^2 \times S^4) \vee S^8)$ is nontrivial. Thus a manifold that is homotopy equivalent must be oriented, since in that case $H^8(M; \mathbb{Z}) \cong H^8((S^2 \times S^4) \vee S^8) \cong \mathbb{Z}$. Oriented closed manifolds satisfy Poincaré Duality. If a closed manifold were to be homotopy equivalent to $(S^2 \times S^4) \vee S^8$, since the latter is path-connected the former better be path-connected. Suppose that $(S^2 \times S^4) \vee S^8$ satisfies Poincaré Duality. Consider the fundamental homology class $[M] \in H_8((S^2 \times S^4) \vee S^8) \cong \mathbb{Z}$. From Poincaré Duality, we know that, for $\alpha \in H^2(M)$, where α is a generator, $[M] \frown \alpha$ generates $H_6(M)$, since $D(\alpha) = [M] \frown \alpha$ is an isomorphism.

Examining the cap product, we have

$$\psi(\sigma \frown \varphi) = \psi(\varphi(\sigma|[v_0, ..., v_k])\sigma|[v_k, ..., v_{k+l}]) \tag{19}$$

$$= \psi(\sigma|[v_0, ..., v_k])\psi(\sigma|[v_k, ..., v_{k+l}]) = (\varphi \smile \psi)(\sigma) \tag{20}$$

Thus, $\psi([M] \frown \alpha) = (\alpha \smile \beta)([M])$, where $\psi \in H^6(M)$ is the generator. Since $[M] \frown \alpha$ is a generator, and ψ is a generator homomorphism, $\psi([M] \frown \alpha)$ is a generator for the ring we are in (here we are using $\mathbb{Z}$). Thus, for $\beta \in H^4((S^2 \times s^4) \vee S^8)$ the generator, $1 = \psi([M] \frown \alpha) =$

$(\alpha \smile \psi)([M]) = (\alpha \smile (\alpha \smile \beta))([M]) = 0([M]) = 0$, a contradiction. This is because $\alpha \smile \alpha = 0$ from the ring structure we derived earlier. Because of this, Poincare Duality is not satisfied, so any orientable or otherwise closed manifold has a different ring structure and therefore is not homotopy equivalent. □

Question 6. *Prove that the Poincaré Duality theorem implies that if F is a field and M^n is a closed $F-$oriented manifold with its fundamental class $[M^n] \in H_n(M^n; F)$, then the pairing*

$$H^k(M^n; F) \times H^{n-k}(M^n; F) \to F \tag{21}$$

$$\phi \times \psi \mapsto \langle \phi \cup \psi, [M^n] \rangle \tag{22}$$

is nonsingular for every $k = 0, ..., n$.

Proof. For F a field, M^n a closed $F-$oriented manifold with fundamental class $[M^n] \in H_n(M^n; F)$, the pairing

$$H^k(M^n; F) \times H^{n-k}(M^n; F) \to F \tag{23}$$

$$\phi \times \psi \mapsto (\phi \smile \psi)([M^n]) \tag{24}$$

is nonsingular if $H^k(M^n; F) \cong \text{Hom}(H^{n-k}(M^n; F), F)$ and $H^{n-k}(M^n; F) \cong \text{Hom}(H^k(M^n; F), F)$.

For the first isomorphism, we want to relate $H^k(M^n; F)$ with $\text{Hom}(H^{n-k}(M^n; F), F)$. Using Poincaré Duality, we have

$$\begin{array}{ccc} H^{n-k}(M^n; F) & \xrightarrow{\cong} & H_k(M^n; F) \\ \downarrow & & \downarrow \\ \text{Hom}(H^{n-k}(M^n; F), F) & \cong & \text{Hom}(H_k(M^n; F), F) \end{array}$$

$\text{Hom}(H^{n-k}(M^n; F), F) \cong \text{Hom}(H_k(M^n; F), F)$ via the hom-dual of Poincaré Duality. We can now relate $H^k(M^n; F)$ with $\text{Hom}(H_k(M^n; F), F)$ through the Universal Coefficient Theorem. The homology groups $H_k(M^n; F)$ are the homology groups of the chain complex of free $F-$modules with basis the singular $n-$simplices in M^n. From the Universal Coefficient Theorem, we have the exact sequence

$$0 \to \text{Ext}(H_{k-1}(M^n; F), F) \to H^k(M^n; F) \to \text{Hom}(H_k(M^n; F), F) \to 0 \tag{25}$$

Since M^n is closed, we have that $H_{k-1}(M^n;F),F)$ is finitely generated. Now we examine $\text{Ext}H_{k-1}(M^n;F),F)$. We wish to define a free resolution of the free $F-$module that is $H_{k-1}(M^n;F),F)$:

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow H_{k-1}(M^n;F) \longrightarrow 0$$

$$0 \longrightarrow 0 \longrightarrow H_{k-1}(M^n;F) \longrightarrow H_{k-1}(M^n;F) \longrightarrow 0$$

Dualizing the last line, we have the exact sequence

$$0 \xleftarrow{f_1} H_{k-1}(M^n;F) \xleftarrow{f_0} H_{k-1}(M^n;F) \longleftarrow 0$$

The definition of Ext is $\text{Ker}(f_1)/\text{Im}(f_0) = H_{k-1}(M^n;F),F)/H_{k-1}(M^n;F),F) = 0$. Thus our exact sequence from the Universal Coefficient Theorem becomes

$$0 \to 0 \to H^k(M^n;F) \to \text{Hom}(H_k(M^n;F),F)) \to 0 \tag{26}$$

Therefore, $H^k(M^n;F) \cong \text{Hom}(H_k(M^n;F),F) \cong \text{Hom}(H^{n-k}(M^n;F),F)$.

That F needs to be a field comes from the first isomorphism above. Denote the free $F-$module $C_k(M^n;F)$ with basis the singular $k-$simplices in M^n. Suppose there are j $k-$simplices in M^n. Then by the Structure Theorem for Principal Ideal Domains, $C_k(M^n;F) \cong F^j$ if F is a field. Thus $\text{Hom}(C_k(M^n;F),F) \cong \text{Hom}(F^j,F) \cong \text{Hom}(C_k(M^n),F)$. Treating $\text{Hom}(C_k(M^n;F),F)$ as a dual complex, the homology groups are the cohomology groups $H^k(M^n;F)$.

We can get the second requirement $H^{n-k}(M^n;F) \cong \text{Hom}(H^k(M^n;F),F)$ the same way, provided that we can take

$$H^{n-k}(M^n;F) \times H^k(M^n;F) \to F \tag{27}$$

$$\psi \times \phi \mapsto (\psi \smile \phi)([M^n]) = (\phi \smile \psi)([M^n]) \tag{28}$$

or, in terms, the cup product commutes. We check that this is true. For a singular $n-$simplex $\sigma : \Delta^n \to M^n$, we have

$$(\phi \smile \psi)([M^n]) = \phi(\sigma|[v_0,...,v_k]) \cdot \psi(\sigma|[v_k,...,v_n]) \tag{29}$$

$$= \psi(\sigma|[v_k,...,v_n]) \cdot \phi(\sigma|[v_0,...,v_k]) = (\psi \smile \phi)([M^n]) \tag{30}$$

since the product in F given by $\cdot$ commutes in a field, and relabeling the vertices of our $n-$simplex. Therefore $H^{n-k}(M^n;F) \cong \text{Hom}(H^k(M^n;F),F)$, and the pairing given by (2) is nonsingular. $\square$

Question 7. *Show that*

$$H_c^*(\mathbb{R}^n;G) \cong \tilde{H}^*(S^n;G) \tag{31}$$

and more generally that if X is a topological space so that in its one-point compactification $X \cup \infty$, the point ∞ has a contractible neighborhood, then

$$H_c^*(X;G) \cong \tilde{H}^*(X \cup \infty;G) \tag{32}$$

where H_c is the cohomology with compact supports.

Proof. In computing $H_c^*(\mathbb{R}^n;G)$, we compute the limit group $\xrightarrow{lim} H^i(\mathbb{R}^n,\mathbb{R}^n-K;G)$, for K compact subsets $K \subset \mathbb{R}^n$. We let each compact subset K be the ball B_k of integer radius k. This is a compatible choice because the integers are a directed set, and any compact subset of $\mathbb{R}^n$ can be contained in a ball of some integer radius. We then use the exact sequence that comes with relative cohomology:

$$H^0(\mathbb{R}^n,\mathbb{R}^n-B_k;G) \to H^0(\mathbb{R}^n;G) \to H^0(\mathbb{R}^n-B_k;G) \to \tag{33}$$

$$H^1(\mathbb{R}^n,\mathbb{R}^n-B_k;G) \to H^1(\mathbb{R}^n;G) \to H^1(\mathbb{R}^n-B_k;G) \to ... \tag{34}$$

$$H^i(\mathbb{R}^n,\mathbb{R}^n-B_k;G) \to H^i(\mathbb{R}^n;G) \to H^i(\mathbb{R}^n-B_k;G) \to ... \tag{35}$$

Since $\mathbb{R}^n$ is simply connected, $H^0(\mathbb{R}^n;G) \cong G$, and $H^i(\mathbb{R}^n;G) \cong 0, i > 0$. Examining $H^0(\mathbb{R}^n,\mathbb{R}^n-B_k;G)$, we see that this is given by $\text{Hom}(C_0(\mathbb{R}^n)/C_0(\mathbb{R}^n-B_k),G)$. Since both $\mathbb{R}^n$ and $\mathbb{R}^n-B_k$ are connected, this group is trivial. Therefore, our exact sequence becomes

$$0 \to G \to H^0(\mathbb{R}^n-B_k;G) \to \tag{36}$$

$$H^1(\mathbb{R}^n,\mathbb{R}^n-B_k;G) \to 0 \to H^1(\mathbb{R}^n-B_k;G) \to ... \tag{37}$$

$$0 \to H^i(\mathbb{R}^n-B_k;G) \to H^{i+1}(\mathbb{R}^n,\mathbb{R}^n-B_k;G) \to 0 \to ... \tag{38}$$

Notice that $\mathbb{R}^n - B_k$ is homotopically equivalent to S^n. Therefore we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow G \longrightarrow & \mathrm{Hom}(C_0(S^n),G) & \longrightarrow & ... & \longrightarrow & \mathrm{Hom}(C_i(S^n),G) & \longrightarrow & ... \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \tilde{H}^0(S^n;G) & \longrightarrow & ... & \longrightarrow & \tilde{H}^i(S^n;G) & \longrightarrow & ... \\
 & & \downarrow \cong & & & & \downarrow \cong & & \\
0 & \longrightarrow & H^0(\mathbb{R}^n,\mathbb{R}^n - B_k;G) & \longrightarrow & ... & \longrightarrow & H^i(\mathbb{R}^n,\mathbb{R}^n - B_k;G) & \longrightarrow & ...
\end{array}$$

Therefore $H^i(\mathbb{R}^n, \mathbb{R}^n - B_k; G) \cong \tilde{H}^i(S^n; G)$. Since $\mathbb{R}^n - B_k \simeq S^n \simeq \mathbb{R}^n - B_{k+1}$, we have that $H^i(\mathbb{R}^n, \mathbb{R}^n - B_k; G) \cong H^i(\mathbb{R}^n, \mathbb{R}^n - B_{k+1}; G)$. Thus, $\xrightarrow{lim} H^i(\mathbb{R}^n, \mathbb{R}^n - B_k; G) \cong H^i_c(\mathbb{R}^n; G) \cong \tilde{H}^i(S^n; G)$. Because of this homotopy equivalence, these cohomology groups must have isomorphic ring structure, so $H^*_c(\mathbb{R}^n; G) \cong \tilde{H}^*(S^n; G)$.

More generally, for a topological space X such that the one-point compactification $X \cup \infty$ has a neighborhood of $\{\infty\}$ that is contractible, we examine the compactly supported cohomology of X. Therefore, there exists a contractible open set $U \subset X \cup \infty$ containing ∞. The compact subsets $K \subset X$ form a directed set under inclusion, since the union of two compact sets is compact. We have $\lim_{\rightarrow} H^i(X, X - K; G) = H^i_c(X; G)$. Let H be the complement of U in $X \cup \infty$. Since U is open, H is closed. Since $H \subset X \cup \infty$, and $X \cup \infty$ is compact, H is bounded. Therefore H is compact. Due to excision, since $\infty \in U \subset X \cup \infty$ has closure in U, we have

$$H^i(X, X - H) \cong H^i(X \cup \infty, X - H \cup \infty) = H^i(X \cup \infty, U \cup \infty) = H^i(X \cup \infty, U) \tag{39}$$

for all i. Since U is contractible, $\lim_{\rightarrow} H^i(X, X - K) = H^i(X, X - H)$. This is because U can contract to a smaller open neighborhood, encompassing any larger compact subset of $X \cup \infty$. We now examine $H^*(X \cup \infty, U; G)$. We have

$$\begin{array}{cccccccc}
 & 0 \longrightarrow & C^0(X\cup\infty;G)/C^0(U;G) & \longrightarrow & C^1(X\cup\infty;G)/C^1(U;G) & \xrightarrow{\cdots} & C^i(X\cup\infty;G)/C^i(U;G)... \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 & 0 \longrightarrow & C^0(X\cup\infty;G)/\mathbb{Z} & \longrightarrow & C^1(X\cup\infty;G) & \xrightarrow{\cdots} & C^i(X\cup\infty;G)... \\
 & & & & \downarrow & & \\
0 \longrightarrow \mathbb{Z} & \longrightarrow & C^0(X\cup\infty;G) & \longrightarrow & C^1(X\cup\infty;G) & \xrightarrow{\cdots} & C^i(X\cup\infty;G)... \\
 & & & & \downarrow & & \\
 & 0 \longrightarrow & \tilde{H}^0(X\cup\infty;G) & \longrightarrow & \tilde{H}^1(X\cup\infty;G) & \xrightarrow{\cdots} & \tilde{H}^i(X\cup\infty;G)...
\end{array}$$

since $C^i(U; G) \cong C^i(\infty; G)$, as U is homotopically equivalent to a point. Thus $H^*(X \cup \infty, U; G) \cong \tilde{H}^*(X \cup \infty; G)$. Since X has the same singular structure of $X \cup \infty$ in $0 <$ dimensions, the ring structure is the same. Thus $H_c^*(X;G) \cong \tilde{H}^*(X \cup \infty; G)$. □

Question 8. *1. A theorem of Hopf states that if X is a path connected space of the homotopy type of a CW-complex, and it is endowed with a basepoint, then there is an isomorphism,*

$$[X, S^1]. \xrightarrow{\cong} H^1(X, \mathbb{Z}) \tag{40}$$

$$f \to f^*(\sigma) \tag{41}$$

where $\sigma \in H^1(S^1;\mathbb{Z}) \cong \mathbb{Z}$ is a generator, and $[X, S^1]$. denotes the based homotopy classes of basepoint preserving maps from X to S^1. Let M^n be a closed, oriented, connected n-dimensional manifold with basepoint $x_0 \in M^n$. Suppose $\alpha \in H^1(M;\mathbb{Z})$. Let $f_\alpha : M \to S^1$ represent α via Hopf's theorem. Let $N = f_\alpha^{-1}(t)$ where $t \in S^1$ is a regular value of f_α. Show that the homology class $[N] \in H_{n-1}(M)$ is Poincaré dual to $\alpha \in H^1(M)$.

2. Prove, using Hopf's theorem, the following theorem of Thom: If M^n is a closed, orientable manifold, then any homology class in $H_{n-1}(M^n)$ is represented by the fundamental class of a smooth codimension one, closed, oriented submanifold.

Proof. 1. Treat $t \in S^1$ as an embedded 0-dimensional submanifold. Since t is a regular value of f_α, $N = f_\alpha^{-1}(t)$ is a submanifold of M of dimension $n-1$ by the Regular Value Theorem (for proof that any α can correspond to a *smooth* map, see part b)). Furthermore, $f \pitchfork t$ since t is 0-dimensional; since t is a regular value, f is submersive at t, and thus has image the entire tangent space $T_t S^1$. We then can invoke the previous problem:

$$[f_\alpha^{-1}(t)] = [N] \in H_{n-1}(M;\mathbb{Z}) \tag{42}$$

$$= f_\alpha^*(D_{S^1}([t])) \cap [M] \tag{43}$$

We now examine $f_\alpha^*(D_{S^1}([t]))$. From the above theorem, we have $[t] \in H_0(S^1;\mathbb{Z})$ the fundamental class of t, up to a sign difference depending on our orientation. Therefore, we have $D_{S^1}([t]) \in H^1(S^1;\mathbb{Z})$ is a generator. Thus $f_\alpha^*(D_{S^1}([t])) = \alpha \in H^1(M;\mathbb{Z})$, up to a sign. Therefore $\alpha \cap [M] = [N]$, i.e. α is Poincaré Dual to $[N]$. (This works out because every manifold is a $CW-$complex, and M is closed, oriented, and connected and thus path-connected)

2. We use the previous part for inspiration. Suppose we have a class $\beta \in H_{n-1}(M)$. Take its Poincaré Dual $D_M(\beta) \in H^1(M)$. In the same way as in part a), let f_β be a smooth map from M to S^1. First we check that f_β is in the same homotopy class as a smooth map. From the Whitney Embedding theorem, we call the smooth embedding $g : S^1 \to \mathbb{R}^2$. We want $g^{-1} \circ g \circ f : M \to S^1$ to be homotopic to a smooth map, so $g \circ f : M \to \mathbb{R}^2$ has to be homotopic to a smooth map. This is a standard fact in analysis: for $\epsilon > 0$, there exists a differentiable function h such that, when we divide up our map $(g \circ f) := \sum_i^n (g \circ f)_i : M \to \mathbb{R}$, we have $|(g \circ f)_i - h| < \epsilon$; the graph of $(g \circ f)_i$ is a continuous section of the trivial bundle $M \times \mathbb{R}$. In any $\epsilon-$neighborhood of $(g \circ f)_i$ there is a differentiable section h. This is the h we want. Since the $\epsilon-$neighborhood is continuously mapped to $(g \circ f)_i$, h homotopic to $(g \circ f)_i$. Thus $(g \circ f)$ (the whole map $M \to \mathbb{R}^2$) maps to a tubular neighborhood of S^1. Since ηS^1 can be smoothly deformed to S^1 (call this map π), we have a differentiable map $\pi \circ (g \circ f) : M \to S^1 \subset \mathbb{R}^2$ that is homotopic to f. From Corollary 8.5 in the notes, the set of regular values of f_β is residual, so there exists a regular value $t' \in S^1$ of this map. Through the Regular Value Theorem, we have $f_\beta^{-1}(t')$ is a submanifold of M. This is oriented because, for $U \subset S^1$ an open subset containing $t \in S^1$, $f_\beta^{-1}(U)$ is an open subset of M and therefore orientable via f_β^{-1}. By similar reasoning, $f_\beta^{-1}(t)$ is closed. From the previous part, $D_M([f_\beta^{-1}(t')]) = D_M(\beta) \in H^1(M)$, i.e. $[f_\beta^{-1}(t')] = \beta \in H^1(M)$. Thus every homology class in $H_{n-1}(M)$ is represented by the fundamental class of a codimension 1, closed, orientable submanifold.

□

1 Bundle Theory

All problems written by Prof. Ralph Cohen. Referenced to Cohen's notes/textbook-in-progress, "Bundles, Homotopy, and Manifolds."

Question 9. *Let $\xi \to B$ be an n-dimensional vector bundle.*

1. *Define clutching functions of the $nk-$dimensional $k-$fold tensor product bundle $\otimes^k \xi \to B$ in terms of clutching functions of ξ.*

2. *Define clutching functions of the $k-$fold exterior product bundle $\wedge^k\xi \to B$ in terms of clutching functions of ξ.*

Proof. 1. Let $\xi \to B$ be an $n-$dimensional vector bundle with clutching functions $\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$. We define clutching functions on the $k-$fold tensor bundle $\otimes^k\xi \to B$ by taking the product of our clutching functions on ξ:

$$\phi^{\otimes^k\xi}_{\alpha,\beta} : \qquad U_\alpha \cap U_\beta \xrightarrow{\phi_{\alpha,\beta}\times\ldots\times\phi_{\alpha,\beta}} GL_n(\mathbb{R}) \times \ldots \times GL_n(\mathbb{R}) \xrightarrow{\quad\otimes\quad} GL_{n^k}(\mathbb{R})$$

$$x \longmapsto^{\phi_{\alpha,\beta}\times\ldots\times\phi_{\alpha,\beta}} A \times \ldots \times A \longmapsto^{\otimes} A \otimes \ldots \otimes A$$

where $A \in GL_n(\mathbb{R})$ is the linear transformation on ξ is the image of the regular clutching function on ξ. Here the tensor product of two linear transformations $A_1 : \xi \to \xi, \ldots, A_k : \xi \to \xi$ is the induced linear transformation $A_1 \otimes \ldots \otimes A_k : \xi \otimes \ldots \otimes \xi \to \xi \otimes \ldots \otimes \xi$. This well-defined because we simply take the automorphism associated to the clutching function on ξ and tensor k copies of it; we get $\phi_{\beta,\alpha} = A^{-1} \otimes \ldots \otimes A^{-1}$, which, when applied before or after $\phi_{\alpha,\beta}$, we get $Id \otimes \ldots \otimes Id$. Thus $\phi_{\alpha,\beta} = \phi^{-1}_{\beta,\alpha}$.

2. We approach this problem by considering the vector space associated with $\otimes^k\xi$, for ξ and $n-$dimensional vector space. Consider an orthonormal basis $\{e_i\}_{0\leq i\leq n}$ of ξ. We define an isomorphism from $\xi \times \ldots \times \xi$ to $\xi \otimes \ldots \otimes \xi$:

$$e_{i_1} \times \ldots \times e_{i_k} \mapsto e_{i_1} \otimes \ldots \otimes e_{i_k} \tag{44}$$

Thus we have nk generators of $\otimes^k\xi$, so $\otimes^k V$ is isomorphic to an $nk-$dimensional vector space. Since $\wedge^k\xi$ a quotient of this vector space, this should probably be isomorphic to a subspace. We construct a basis of $\wedge^k\xi$. The symmetry quotient $a\otimes b + b\otimes a$ implies that $a \otimes b = -b \otimes a$, in particular $a \otimes a = -a \otimes a$, so $a \otimes a = 0$, for $a, b \in \xi$. Thus there can be no repeated indices, and permutations of index combinations are linearly dependent. Thus, the allowable basis vectors are just the $\binom{n}{k}$ combinations of k entries spanning 1 to n. Thus $\wedge^k\xi$ is isomorphic to an $\binom{n}{k}$-dimensional vector space. If $k > n$, there must be repeated indices, and so the vector space is 0-dimensional. Thus we need our clutching functions to have image in $GL_{\binom{n}{k}}(\mathbb{R})$ from k elements in $GL_n(\mathbb{R})$. Antisymmetry, along with general objects one runs into with dealing with exterior algebras, are determinants. Suppose $[g_{ij}] \in GL_k(\mathbb{R})$ is the transition function associated with $U_i \cap U_j$, and associate the $k \times k$-submatrix of $[g_{ij}]$ denoted $[g_{ij}]_{h \in \binom{n}{k}}$ (henceforth we will say that $h \in S(\binom{n}{k})$ is an increasing sequence of k integers between 1 and n), with entries kept being the entries associated with the corresponding indices of the basis vectors of $\wedge^k \xi$. Define the element in $GL_{\binom{n}{k}}(\mathbb{R})$ as the matrix

$$([g_{ij}]_{h \in S(\binom{n}{k})})_{m,l \in \binom{n}{k}} \tag{45}$$

Example 1.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \mapsto \begin{pmatrix} \begin{pmatrix} a & b \\ d & e \end{pmatrix} & \begin{pmatrix} a & c \\ d & f \end{pmatrix} & \begin{pmatrix} b & b \\ e & f \end{pmatrix} \\ \begin{pmatrix} a & b \\ g & h \end{pmatrix} & \begin{pmatrix} a & c \\ g & i \end{pmatrix} & \begin{pmatrix} b & c \\ h & i \end{pmatrix} \\ \begin{pmatrix} d & e \\ g & h \end{pmatrix} & \begin{pmatrix} d & f \\ g & i \end{pmatrix} & \begin{pmatrix} e & f \\ h & i \end{pmatrix} \end{pmatrix} \tag{46}$$

where the submatrix entries are determined by ranging across the elements of the $\binom{n}{k}$ *permutations.*

This is clearly an injection, as scaling any entry in the preimage scales a unique combination of entries in the image. Thus, by changing one entry in the image, one cannot counteract this change by changing another entry. Furthermore, Id_n trivially maps to $Id_{\binom{n}{k}}$, since all off-diagonal elements have the determinants of matrices with only one nonzero element, and the diagonal elements have determinants of identity matrices. Thus the image is in a subgroup of $GL_{\binom{n}{k}}(\mathbb{R})$, and this map has a well-defined inverse. To check that this is a well-defined clutching function, we consider the clutching function associated with $[g_{ij}^{-1}]$. We have

$$A_{ij}^{-1} = \frac{n! \epsilon^{i_1 \ldots i_b} \epsilon_{j_1 \ldots j_b} A_{i_1}^{j_1} \ldots A_{i_b}^{j_b}}{k! \epsilon^{i i_2 \ldots i_b} \epsilon_{j j_2 \ldots j_b} A_{i_2}^{j_2} \ldots A_{i_b}^{j_b}}, \text{ for } b := \frac{n!}{k!} \tag{47}$$

$$= \frac{n \epsilon^{i_1 \ldots i_b} \epsilon_{j_1 \ldots j_b} [[g_{ij}]_{h_1}] \ldots [[g_{ij}]_{h_b}]}{\epsilon^{i i_2 \ldots i_b} \epsilon_{j j_2 \ldots j_b} [[g_{ij}]_{h_2}] \ldots [[g_{ij}]_{h_b}]}, h_\alpha \in S(\binom{n}{k}). \tag{48}$$

where h_α is the corresponding permutation with the i_α, j_α entry in our matrix.

$$(g_{ij})^{-1} = \frac{n\epsilon^{i_1 \dots i_n}\epsilon_{j_1 \dots j_n} g_{i_1 j_1} \dots g_{i_n j_n}}{\epsilon^{i i_2 \dots i_n}\epsilon_{j j_2 \dots j_n} g_{i_2 j_2} \dots g_{i_n j_n}} \tag{49}$$

$$((g_{ij}^{-1}))_{h \in S(\binom{n}{k})})_{m,l \in \binom{n}{k}} = \frac{n\epsilon^{i_1 \dots i_b}\epsilon_{j_1 \dots j_b}\epsilon^{s_1 \dots t_k}\epsilon_{s_1 \dots t_k} g_{s_1 t_1} \dots g_{s_k t_k}\epsilon^{i''_1 \dots i''_k}\epsilon_{j''_1 \dots j''_k} g_{i''_1 j''_1} \dots g_{i''_k j''_k}}{\epsilon^{i i_2 \dots i_b}\epsilon_{j j_2 \dots j_b}\epsilon^{i'_1 \dots i'_k}\epsilon_{j'_1 \dots j'_k} g_{i'_1 j'_1} \dots g_{i'_k j'_k}\epsilon^{i''_1 \dots i''_k}\epsilon_{j''_1 \dots j''_k} g_{i''_1 j''_1} \dots g_{i''_k j''_k}} \tag{50}$$

$$= \frac{n\epsilon^{i_1 \dots i_b}\epsilon_{j_1 \dots j_b}[[g_{ij}]_{h_1}] \dots [[g_{ij}]_{h_b}]}{\epsilon^{i i_2 \dots i_b}\epsilon_{j j_2 \dots j_b}[[g_{ij}]_{h_2}] \dots [[g_{ij}]_{h_b}]}, h_\alpha \in S(\binom{n}{k}) \tag{51}$$

where in the last step we rewrite our scare labels for elements of $S(\binom{n}{k})$, and we find this is exactly equal to A_{ij}^{-1}. Thus A_{ij}^{-1} is the image of g_{ij}^{-1}, and so our clutching functions are well-defined.

□

Question 10. *1. Notice that the tensor product of two one-dimensional vector bundles ("line bundles") over a space B is still a one dimensional vector bundle . Show that the set of isomorphism classes of one-dimensional (real) vector bundles over B is an abelian monoid with respect to tensor product. In particular, what is the unit of this monoid?*

2. Show that in fact this abelian monoid is an abelian group.

Proof. 1. First we note that the clutching functions of a vector bundle uniquely determine the isomorphism class of said bundle. In the spirit of the first problem, we define our clutching functions for the $k-$fold tensor product of line bundles by multiplying the clutching functions of a single line bundle:

$$\phi_{\alpha,\beta}^{\otimes^k \mathbb{R}} : U_\alpha \cap U_\beta \xrightarrow{\phi_{\alpha,\beta} \times \dots \times \phi_{\alpha,\beta}} \mathbb{R}^* \times \dots \times \mathbb{R}^* \xrightarrow{\cong} \mathbb{R} \tag{52}$$

$$\to \mathbb{R} \otimes \dots \otimes \mathbb{R} \tag{53}$$

where that last isomorphism is due to having the basis vector, comprised of the e_i basis vector for the i^{th} tensor factor, as $e_1 \otimes \dots \otimes e_k^*$. This is an associative operation, as $s_{\alpha,\beta}(\tilde{s}_{\alpha,\beta} s'_{\alpha,\beta}) \to (e_1 \otimes e_2) \otimes e_3 \cong e_1 \otimes (e_2 \otimes e_3) \leftarrow (s_{\alpha,\beta}\tilde{s}_{\alpha,\beta})s'_{\alpha,\beta}$. This map is abelian, since we can simply map $e_i \times e_j \to e_j \otimes e_i$, as an isomorphism, and as $e_i \times e_j \to e_i \otimes e_j$ is an isomorphism, $\otimes \mathbb{R}' \cong \mathbb{R}' \otimes \mathbb{R}$. Thus these are the same isomorphism class. It remains to show that there is an identity element. From the properties of the tensor product, $e_1 \otimes e_2 = e_1 e_2 \otimes 1 = 1 \otimes e_1 e_2$. Furthermore, $s_{\alpha,\beta} s_{\beta,\alpha} = 1$. Thus we have a natural isomorphism from principal bundles $(B \times \{1\} = B)$ $\mathbb{R} \to \mathbb{R} \otimes \mathbb{R}$ given by $e_1 \to e_1 \otimes 1 \cong 1 \otimes e_1$. Using the inverse map (that map was an isomorphism) Thus the identity element is given by $1 \in \mathbb{R}^*$. Since we have associativity, commutativity, and an identity element, the isomorphism classes of 1-dimensional vector bundles forms an abelian monoid.

2. It remains to prove that every element has a unique inverse, as an abelian group is an abelian monoid where every element has a unique inverse. Consider the map

$$\phi^{\otimes^2 \mathbb{R}}_{\alpha,\alpha} : U_\alpha \cap U_\beta \xrightarrow{\phi_{\alpha,\beta} \times \phi_{\beta,\alpha}} \mathbb{R}^* \times \mathbb{R}^* \xrightarrow{\otimes} \mathbb{R} \otimes \mathbb{R} \xrightarrow{\cong} \mathbb{R} \tag{54}$$

$$x \to a \times a^{-1} \to a \otimes a^{-1} \xrightarrow{b} aa^{-1} \otimes 1 \to 1 \otimes 1 \xrightarrow{c} 1 \tag{55}$$

where the b map is due to the linearity of the tensor product and the c map is due to the natural isomorphism defined above. Since $\phi_{\beta,\alpha}$ is the unique inverse clutching function to $\phi_{\alpha,\beta}$, each element in this monoid has a unique inverse, and thus the isomorphism classes of 1-dimensional vector bundles is an abelian group with the tensor product operation.

□

Question 11. *Let X be a space with a basepoint $x_0 \in X$. Recall that the (reduced) suspension of X, ΣX, is the space*

$$\Sigma X = X \times S^1 / \{X \times \{1\} \cup x_0 \times S^1\} \tag{56}$$

Here I am thinking of S^1 as the unit complex numbers. Let $(Y, y0)$ be another space with basepoint. Consider the (based) "loop space"

$$\Omega Y = Map((S^1, \{1\}), (Y, y_0)) \tag{57}$$

This is the space of maps from S^1 to Y that take $1 \in S^1$ to the basepoint $y_0 \in Y$, endowed with the compact - open topology.

1. *Prove that there is a bijection*

$$[\Sigma X, Y] \cong [X, \Omega Y] \tag{58}$$

Here the notation [−, −] denotes the set of homotopy classes of basepoint preserving maps. As a special case, conclude that $\pi_n(Y, y_0) \cong \pi_{n-1}(\Omega Y, \epsilon_0)$, where $\epsilon_0 : S^1 \to Y$ is the constant map at the basepoint y_0.

2. *Let G be a topological group, and consider the map $f : G \to \Omega BG$ defined in the proof of Corollary 4.10 in the text. Prove that f induces an isomorphism in homotopy groups (in all degrees). Such a map is called a "weak homotopy equivalence".*

Proof. 1. We being by considering elements of $[X, \Omega Y]$. Define a basepoint-preserving map $f \in [X, \Omega Y]$ for some $x \in X$ by $f(x)$. This is thus a basepoint-preserving map from $(S^1, \{1\})$ to (Y, y_0), denoted $f(x)(t)$. Since $f(x)(t)$ is basepoint-preserving, $f(x)(1) = y_0$. Furthermore, since f is basepoint-preserving, $f(x_0)(t) = y_0$. We now examine elements of $[\Sigma X, Y]$. Suppose we have a basepoint-preserving map $g \in [\Sigma X, Y]$. If we consider $(x, t) \in X \times S^1/\{X \times \{1\} \cup x_0 \times S^1\}$, we have that $(x, 1) = (x_0, t), \forall x \in X, t \in S^1$. Thus we have $g(x, 1) = g(x_0, t) = y_0$. We have our correspondence as $[g(x, t)] \mapsto [f(x)(t)]$, given by the corresponding that, for $g : \Sigma X \to Y$, we associate the family of loops $f(x)(t)$ by restricting g to the images of the loops $\{x\} \times S^1 \subset \Sigma X$. We first prove surjectivity. Given a map $j : (S^1, \{1\}) \to (Y, y_0)$, we have the preimage of this map in $[X, \Omega Y]$ to be the $x \in X$ such that $g(x, t) \subset Y$ is of the same homotopy type in Y (we fix x and let t span S^1 to get the same loop). Thus we associate the homotopy type of $g(x, t)$ with the homotopy type of $[f(x)(t)]$, so the map is surjective. Suppose $[g_1(x_1, t_1)] \neq [g_2(x_2, t_2)]$. Then we associate the maps $[f_1(x_1)(t_1)], [f_2(x_2)(t_2)] \in [X, \Omega Y]$, respectively. Since $x_1 \neq x_2, [g_1] \neq [g_2]$, the maps $f_1(x_1) : S^1 \to Y, f_2(x_2) : S^1 \to Y$ are not homotopically equivalent, and thus the loops are not homotopically equivalent. Thus $[f_1(x_1)(t_1)] \neq [f_2(x_2)(t_2)]$. We check that this works for the basepoint map: $[g(x_0, t)] \mapsto [f(x_0)(t)] = [\text{the constant map}]$.

Notice that $\pi_n(Y, y_0) = [S^n, Y]$, so it suffices to prove that $\Sigma S^{n-1} = S^n$. We do this using

$CW-$complexes:

$$\frac{[e^0 \sqcup e^n/\sim] \times [e^0 \sqcup e^1]/\sim}{[e^0 \sqcup e^n/\sim] \times e^0 \cup [e^0 \sqcup e^1/\sim] \times e^0} = \frac{(e^0 \sqcup e^0) \times (e^0 \times e^1) \times (e^0 \times e^1) \times (e^1 \times e^n)/\sim}{[e^0 \times e^0] \cup [e^0 \times e^1] \cup [e^0 \times e^n]/\sim} \tag{59}$$

$$= e^0 \sqcup (e^n \times e^1)/\sim \tag{60}$$

$$= S^{n+1} \tag{61}$$

where we quotient out the usual way, i.e. the terms in the quotient "cancel" the equal terms in the space and become a single point e^0. Thus we have

$$\pi_{n-1}(\Omega Y, \epsilon_0) \cong [S^{n-1}, \Omega Y] \cong [\Sigma S^{n-1}, Y] \cong [S^n, Y] \cong \pi_n(Y, y_0) \tag{62}$$

2. First we prove injectivity. Since $\overline{f}(g)(t) = f(g, t)$, we notice the basepoint-preserving-ness of $\overline{f}$. For basepoint $g_0 \in G$, we have $\overline{f}(g_0)(t) = f(g_0, t) = f(g_0, t')$, for any $t' \in S^1$, equal to the constant map. Consider a class of a nullhomotopic loop in $\pi_n(G)$. We have $\overline{f}(g_0)(t)$ is the constant map, mapping the point g_0 to ϵ_0, the constant map $S^1 \to BG$. This is because $(g_0, t) = g_0$, so g_0 gets mapped to $\{1\} \in S^1$ which gets constantly mapped to the basepoint of BG, as it is constant for all $t \in S^1$. Thus the identity of $\pi_n(G)$ maps to the identity of $\pi_n(\Omega BG)$. Now we prove surjectivity. Suppose we have a homotopy class of an $n-$dimensional loop $[n] \in \pi_n(\Omega BG)$. This corresponds to an $(n+1)-$dimensional homotopy class of BG through the isomorphism proved above. We seek to create a principal G-bundle over ΣG that is trivial on both cones. Define this bundle as

$$C_+ := G \times [1, -1]/\sim, c_- := G \times [-1, 1] \tag{63}$$

$$E := C_+ \times G \cup_{Id} C_- \times G \tag{64}$$

By theorem 4.8 in the text, there is a bijective correspondence given by

$$\psi : [\Sigma G, BG] \to Prin_G(\Sigma G) \tag{65}$$

$$f \mapsto f^*(E) \tag{66}$$

such that $f^*(EG) = E$. Thus let $[g] \in \pi_{n+1}(\Sigma G)$ be the homotopy class of an $(n+1)-$loop that maps its loop in ΣG to said $(n+1)-$loop in BG induced by f, well-defined because f is a bijection. Thus let g be the $\overline{f}(g)(t)$. We know this must exist due to part a). Therefore, $\overline{f}$ is an isomorphism in π_n. Since this did not depend on n, all such homotopy groups are isomorphic.

□

Question 12. *For any space X let $Vect^d(X)$ denote the set of isomorphism classes of $d-$dimensional vector bundles over X.*

1. *Compute $Vect^d(S^1)$. Justify your answer.*
2. *Compute the fundamental group of the Grassmannian, $\pi_1(Gr_d(\mathbb{R}^\infty)$.*
3. *Let X be a simply-connected space. Prove that any one-dimensional vector bundle over X is trivial.*

Proof.

Lemma 2. *There is a bijective correspondence between principal bundles and homotopy groups $Prin_G(S^n) \cong \pi_{n-1}(G)$ where as a set $\pi_{n-1}(G) = [S^{n-1}, x_0; G, \{1\}]$, which refers to (based) homotopy classes of basepoint preserving maps from the sphere S^{n-1} with basepoint $x_0 \in S^{n-1}$, to the group G with basepoint the identity 1$\in G$.*

Proof. Let $p : E \to S^n$ be a principal $G-$bundle. Write S^n as the union of its upper and lower hemispheres

$$S^n = D^n_+ \cup_{S^{n-1}D^n_-} \tag{67}$$

Since $D^n_\pm$ are contractible, the restiction of E to each of the hemispheres is trivial, so if we fix a trivialization of the fiber of E above $x_0 \in S^{n-1} \subset S^n$, we can extend this trivialization to the upper and lower hemispheres. For θ a clutching function on the equator $\theta : S^{n-1} \to G$, we can then write

$$E = (D^n_+ \times G) \cup_\theta (D^n_- \times G) \tag{68}$$

that is, for $(x, g) \in (D^n_+ \times G)$, we have $(x, g) \sim (x, \theta(x)g) \in (D^n_- \times G)$. Since our original trivializations extended a common trivialization on the basepoint $x_0 \in S^{n-1}$, then the trivialization $\theta : S^{n-1} \to G$ maps the basepoint x_0 to the identity $1 \in G$. The assignment of a bundle its clutching function, will define our correspondence

$$\Theta : Prin_G(S^n) \to \pi_{n-1}(G) \tag{69}$$

To see that this correspondence is well defined we need to check that if E_1 is isomorphic to E_2, then the corresponding clutching functions θ_1 and θ_2 are homotopic. Let $\Psi : E_1 \to E_2$ be an isomorphism. We may assume this isomorphism respects the given trivializations of these fibers of these bundles over the basepoint $x_0 \in S^{n-1} \subset S^n$. Then the isomorphism Ψ determines an isomorphism

$$(D^n_+ \times G) \cup_{\theta_1} (D^n_- \times G) \xrightarrow{\Psi} (D^n_+ \times G) \cup_{\theta_2} (D^n_- \times G) \tag{70}$$

By restricting to the upper and lower hemispheres, Ψ defines maps

$$\Psi_+ : D^n_+ \to G \tag{71}$$

$$\Psi_- : D^n_- \to G \tag{72}$$

which both map $x_0 \in S^{n-1}$ to the identity $1 \in G$, and have the property

$$\Psi_+(x)\theta_1(x) = \theta_2(x)\Psi_-(x) \tag{73}$$

or $\Psi_+(x)\theta_1(x)\Psi_-(x)^{-1} = \theta_2(x) \in G$. By considering the linear homotopy $\Psi_+(tx)\theta_1(tx)\Psi_-(tx)^{-1}$ for $t \in [0,1]$, we can see that $\theta_2(x)$ is homotopic to $\Psi_+(0)\theta_1(x)\Psi_-(0)^{-1}$, for 0 the origin in $D^n_\pm$, i.e. the north and south poles of the sphere. Since $\Psi_\pm$ are defined on connected spaces, their images lie on a connected component of G. Since their image on the basepoint $x_0 \in S^{n-1}$ are both the identity, there exist paths $\alpha_+(t)$ and $\alpha_-(t)$ in S^n that start when $t = 0$ at $\Psi_+(0)$ and $\Psi_-(0)$ respectively, and both end at $t = 1$ at the identity $1 \in G$. Then the homotopy $\alpha_+(t)\theta_1(x)\alpha_-(t)^{-1}$ is a homotopy from the map $\Psi + (0)\theta_1(x)\Psi_-(0)^{-1}$ to the map $\theta_1(x)$. Since the first of these maps is homotopic to $\theta_2(x)$, we have that θ_1 is homotopic to θ_2, as claimed. This implies that the map $\theta : Prin_G(S^n) \to \pi_{n-1}(G)$ is well defined.

The fact that Θ is surjective comes from the fact that every map $S^{n-1} \to G$ can be viewed as the clutching function of the bundle

$$E = (D^n_+ \times G) \cup_\theta (D^n_- \times G) \tag{74}$$

We discuss injectivity. Suppose E_1 and E_2 have homotopic clutching functions, $\theta_1 \simeq \theta_2 : S^{n-1} \to G$. We need to show that E_1 is isomorphic to E_2, where

$$E_i = (D^n_+ \times G) \cup_{\theta_i} (D^n_- \times G) \tag{75}$$

Let $H : S^{n-1} \times [-1,1] \to G$ be a homotopy so that $H_1 = \theta_1$ and $H_1 = \theta_2$. Identify the closure of an open neighborhood $\mathcal{N}$ of the equator $S^{n-1} \subset S^n$ with $S^{n-1} \times [-1,1]$. Write $\mathcal{D}_+ = D^2_+ \cup \overline{\mathcal{N}}$ and $\mathcal{D}_- = D^2_- \cup \overline{\mathcal{N}}$. Then $\mathcal{D}_+$ and $\mathcal{D}_-$ are topologically closed disks and hence contractible, with

$$\mathcal{D}_+ \cap \mathcal{D}_- = \overline{\mathcal{N}} \cong S^1 \times [-1,1] \tag{76}$$

Thus we may form the principal $G-$bundle

$$E = \mathcal{D}_+ \times G \cup_H \mathcal{D}_+ \times G \tag{77}$$

where, by abuse of notation, H is the composition $\overline{\mathcal{N}} \cong S^{n-1} \times [-1,1] \xrightarrow{H} G$. If we deformation retract $\overline{\mathcal{N}}$ to S^{n-1} and contract $D^2_\pm$ to D_-, we get that E is isomorphic to E_1 and E_2. $\square$

Lemma 3. *There are bijective correspondences*

$$Vect^1(X) \cong Prin_{U(1)}(X) \cong [X, BU(1)] = [X, \mathbb{C}P^\infty] \cong [X, K(\mathbb{Z}, 2)] \cong H^2(X; \mathbb{Z}) \tag{78}$$

Similarly, there are bijective correspondences

$$Vect^1_{\mathbb{R}}(X) \cong Prin_{O(1)}(X) \cong [X, BO(1)] = [X, \mathbb{R}P^\infty] \cong [X, K(\mathbb{Z}_2, 1)] \cong H^1(X; \mathbb{Z}_2) \tag{79}$$

Proof. The last correspondence takes a map $f : X \to \mathbb{C}P^\infty$ to the class

$$c_1 = f^*(c) \in H^2(X; \mathbb{Z}) \tag{80}$$

where $c \in H^2(\mathbb{C}P^\infty)$ is the generator. In the composition of these correspondences, the class $c_1 \in H^2(X)$ corresponding to a line bundle $\zeta \in Vect^1(X)$ is called the first Chern class of ζ (or of the corresponding principal $U(1)-$bundle). These other correspondences follow directly from the above considerations, once we recall that $Vect^1(X) \cong Prin_{GL(1,\mathbb{C})}(X)\mathbb{C}[X, BGL(1, \mathbb{C})]$, and that $\mathbb{C}P^\infty$ is a model for $BGL(1, \mathbb{C})$ as well as $BU(1)$. This is because we can express $\mathbb{C}P^\infty$ in its homogeneous form as $\mathbb{C}P^\infty = \lim_{n\to\infty}(\mathbb{C}^{n+1} - \{0\})/GL(1, \mathbb{C})$, and that $\lim_{n\to\infty}(\mathbb{C}^{n+1} - \{0\})$ is an aspherical space with a free action of $GL(1, \mathbb{C}) = \mathbb{C}^*$.

For the other case, we have the last correspondence taking a map $f : X \to \mathbb{R}P^\infty$ to the class $\omega_1 = f^*(\omega) \in H^1(X; \mathbb{Z}_2$, where $\omega \in H^1(\mathbb{R}P^\infty; \mathbb{Z}_2)$ is the generator. In the composition of these correspondences, the class $\omega_1 \in H^1(X; \mathbb{Z}_2)$ corresponding to a line bundle $\zeta \in Vect^1_{\mathbb{R}}(X)$ is called the first Stiefel-Whitney class of ζ (or of the corresponding principal $O(1)-$ bundle). □

1. Let $V_d(\mathbb{R}^n)$ be the Stiefel manifold as in the text. We claim that the inclusion of $\mathbb{R}^n$ into $\mathbb{R}^{2n}$ to the first n coordinates induces a nullhomotopic inclusion of $V_d(\mathbb{R}^n)$ into $V_d(\mathbb{R}^{2n})$. Let $\iota : \mathbb{R}^n \to \mathbb{R}^{2n}$ be a linear embedding with image the last n coordinates in $\mathbb{R}^{2n}$. For any $\rho \in V_d(\mathbb{R}^n) \subset V_d(\mathbb{R}^{2n})$, we have a homotopy $t\iota + (1-t)\rho$ that defines a one-parameter family of linear embeddings of $\mathbb{R}^n$ into $\mathbb{R}^{2n}$, and hence a contraction of the image in $V_d(\mathbb{R}^n)$ onto the element ι. Hence the limiting space $V_d(\mathbb{R}^\infty)$ is aspherical with a free $GL(d, \mathbb{R})-$action.

Therefore the projection

$$V_d(\mathbb{R}^\infty) \to V_d(\mathbb{R}^\infty)/GL(d,\mathbb{R}) = Gr_d(\mathbb{R}^\infty) \tag{81}$$

is a universal $GL(d,\mathbb{R})-$bundle, so the infinite Grassmannian is the classifying space $Gr_d(\mathbb{R}^\infty) = BGL(d,\mathbb{R})$, so we have a classification

$$Vect^d(S^1) \cong Prin_{GL(d,\mathbb{R})}(S^1) \cong [S^1, BGL(d,\mathbb{R})] \cong [S^1, Gr_d(\mathbb{R}^\infty)] = \pi_1(Gr_d(\mathbb{R}^\infty)) \tag{82}$$

Thus it remains to compute $\pi_1(Gr_d(\mathbb{R}^\infty))$. Let $V_d^O(\mathbb{R}^n)$ be the Stiefel manifold of orthonormal $d-$frames in $\mathbb{R}^n$. Let $\iota' : \mathbb{R}^n \to \mathbb{R}^{2n}$ be a linear embedding with image an orthonormal frame in the last n coordinates in $\mathbb{R}^{2n}$. For any $\rho' \in V_d^O(\mathbb{R}^n) \subset V_d^O(\mathbb{R}^{2n})$, we have a homotopy $t\iota' + (1-t)\rho'$ that defines a one-parameter family of linear embeddings of $\mathbb{R}^n$ into $\mathbb{R}^{2n}$, and hence a contraction of the image in $V_d^O(\mathbb{R}^n)$ onto the element ι'. Hence the limiting space $V_d^O(\mathbb{R}^\infty)$ is aspherical with a free $O(d)-$action. Therefore the projection

$$V_d^O(\mathbb{R}^\infty) \to V_d^O(\mathbb{R}^\infty)/O(d) = Gr_d(\mathbb{R}^\infty) \tag{83}$$

is a universal $O(d)-$bundle, so the infinite Grassmannian is the classifying space $Gr_d(\mathbb{R}^\infty) = BO(d)$. Thus we have

$$\pi_1(Gr_d(\mathbb{R}^\infty)) \cong \pi_1(BO(d)) \cong [S^1, BO(d)] \cong Prin_{O(d)}(S^1) \xrightarrow{x} \pi_0(O(n)) \tag{84}$$

where the last x map is a bijection due to Lemma 1. $O(d)$ has two connected components: the map is a polynomial and thus continuous. This maps $O(d)$ to either 1 or -1. Therefore, since $\pi_0(O(d))$ is the set of connected components $O(d)$, it is a two-element group, and is thus $\mathbb{Z}/2\mathbb{Z}$. Thus $Vect^d(S^1) \cong \pi_1(Gr_d(\mathbb{R}^\infty)) \cong \mathbb{Z}/2\mathbb{Z}$.

2. See part a).

3. From Theorem 4.14, we have

$$Vect^1(X) \cong Prin_{O(1)}(X) \cong [X, BO(1)] = [X, \mathbb{R}P^\infty] \cong [X, K(\mathbb{Z}_2, 1)] \cong H^1(X; \mathbb{Z}_2) \qquad (85)$$

Since X is simply connected, $H^1(X; \mathbb{Z}_2) = 0$, and so there is only one element in $Vect^1(X)$, i.e. there is only one isomorphism class of 1-dimensional vector bundles, which must be the trivial bundle.

□

Question 13. *Let T^2 be a closed, connected, orientable surface (two-dimensional manifold). Show that there are infinitely many nonisomorphic complex line bundles over T^2.*

Proof. From Theorem 4.13 in the book, we have

$$Vect^1(T) \cong Prin_{U(1)}(T) \cong [T, BU(1)] = [T, \mathbb{C}P^\infty] = [T, K(\mathbb{Z}, 2)] \cong H^2(T, \mathbb{Z}) \qquad (86)$$

Since T is closed and orientable, we can apply Poincaré Duality. Due to Poincaré Duality, we have $H_2(T, \mathbb{Z}) \cong H_0(T)$. Since T is connected, we want to show that T is path-connected because it's a manifold. Let $x \in T$ be any point in T. Let U denote the open neighborhood of x with is locally path-connected. This is assured because there exists an open neighborhood of x that is homeomorphic to $\mathbb{R}^2$, which is everywhere path-connected. Let $y \in T \backslash U$. There exists an open neighborhood of y that is path-connected by the same argument. Thus $U, T\backslash U$ are open, and $U \cup T\backslash U = T$. Since U is nonempty, T must be path-connected, and the boundary of any image of a singular chain is homotopic to the boundary of a point (since every point can be homotoped to any other point), and is thus zero. Thus we have

$$C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} 0 \Rightarrow \qquad (87)$$

$$H_0(T) = \frac{C_0}{Im(\partial_1)} \qquad (88)$$

$$\varepsilon : C_0 \to \mathbb{Z} \qquad (89)$$

$$\varepsilon(\sum_i n_i \sigma_i) \mapsto \sum_i n_i \qquad (90)$$

This is obviously a surjective map since T is nonempty. We want to show that $\text{Ker}(\varepsilon) = \text{Im}(\partial)$. For a 1-simplex $\sigma : \Delta^1 \to X$, we have $\varepsilon(\partial_1(\sigma)) = \varepsilon(\sigma|[v_1] - \sigma|[v_0]) = 1 - 1 = 0$, so $\text{Im}(\partial) \subset \text{Ker}(\varepsilon)$. Now suppose $\varepsilon(\sum_i n_i\sigma_1) = 0$. The σ_i's are singular 0-simplices i.e. points of T. Choose a path $f_t : [0,1] \to T$ from a basepoint x_0 to $\sigma_i(v_0)$, with σ_0 the singular 0-simplex with image x_0. f_t is a singular 1-simplex, and $\partial f_t = \sigma_i - \sigma_0$. Thus $\partial(\sum_i n_i f_t) = \sum n_i\sigma_i - \sum_i n_i\sigma_0 = \sum_i n_i\sigma_i - 0$. Therefore $\sum_i n_i\sigma_i$ is a boundary. Thus $\text{Im}(\partial) \subset \text{Ker}(\varepsilon)$, so $\text{Ker}(\varepsilon) = \text{Im}(\partial)$, and thus $H_0(T) \cong \mathbb{Z}$. This has infinitely many elements, so $H^2(T, \mathbb{Z}) \cong \mathbb{Z}$ has infinitely many elements, so $Vect^1(T)$ has infinitely many elements i.e. isomorphism classes of complex line bundles. □

Question 14. *A vector bundle η is said to be stably trivial if for some $k \in \mathbb{Z}$, the Whitney sum $\eta \oplus \epsilon^k$ is a trivial vector bundle, where ϵ^k denotes the standard trivial bundle of dimension k. Let M be an $n-$dimensional smooth, closed manifold, and suppose that there exists an immersion*

$$f : M \times \mathbb{R}^k \to \mathbb{R}^{n+k} \tag{91}$$

1. *Prove that the tangent bundle TM is stably trivial.*
2. *Show that the sphere S^n has stably trivial tangent bundle for every n. (A manifold with stably trivial tangent bundle is called "stably parallelizable".)*
3. *Show that the tangent bundle $TS^2 \to S^2$ is not trivial, but $TS^2 \oplus \epsilon^1$ is trivial.*

Proof. 1. Since we have an immersion

$$f : M \times \mathbb{R}^k \to \mathbb{R}^{n+k} \tag{92}$$

we have a monomorphism

$$\begin{array}{ccc} T(M \times \mathbb{R}^k) & \xrightarrow{Df} & T\mathbb{R}^{n+k} \\ \downarrow & & \downarrow \\ M \times \mathbb{R}^k & \xrightarrow{f} & \mathbb{R}^{n+k} \\ \downarrow{\scriptstyle \pi} & & \\ M & & \end{array} \qquad\qquad \begin{array}{ccc} \zeta & \xrightarrow{\overline{\gamma}} & \xi \\ \downarrow & & \downarrow \\ X & \xrightarrow{\gamma} & Y \end{array}$$

so that $\gamma_x : \zeta_x \to \xi_{\gamma(x)}$ is a monomorphism of fibers. Since M is $n-$dimensional, since γ_x is injective, it must be an isomorphism as well. Thus we have an isomorphism of vector bundles

$T(M \times \mathbb{R}^k) \cong f^*(T\mathbb{R}^{n+k})$. We have isomorphisms

$$T(M \times \mathbb{R}^k) \cong \pi^*(TM) \oplus \epsilon^k,\ T\mathbb{R}^{n+k} \cong \epsilon^{n+k} \tag{93}$$

$$(v, e) \mapsto v \oplus e,\ (w) \mapsto w \tag{94}$$

Thus we have an isomorphism $\pi^*(TM) \otimes \epsilon^k \cong \epsilon^{n+k}$ of vector bundles over $M \times \mathbb{R}^k$. The pullback of this along a section of π yields $TM \oplus \epsilon^k \cong \epsilon^{n+k}$.

2. We consider the standard embedding $f : S^n \to \mathbb{R}^{n+1}, f(x_1, ..., x_{n+1}) = (x_1, ..., x_{n+1})$. This is obviously an embedding and thus an immersion. The unit normal vector in this embedding is $\frac{\mathbf{x}}{|\mathbf{x}|}$ with respect to the usual euclidean metric. This is nowhere-vanishing on S^n, so the normal bundle is given by $t\frac{\mathbf{x}}{|\mathbf{x}|}, t \in \mathbb{R}$. We have the isomorphism from the trivial line bundle ϵ^1 to the normal bundle by $v \mapsto v\frac{\mathbf{x}}{|\mathbf{x}|}$. Thus we have $TS^n \oplus \nu(S^n) \cong TS^n \oplus \epsilon^1 = \mathbb{R}^{n+1} \cong \epsilon^{n+1}$. Thus TS^n is a stably trivial bundle for all n.

3. From part b) we know that $TS^2 \oplus \epsilon^1 \cong \epsilon^{2+1}$, so $TS^2 \oplus \epsilon^1$ is trivial, so it remains to show that $TS^2 \to S^2$ is not trivial. If TS^2 was nontrivial, there would exist nowhere vanishing sections on S^2, i.e. a nowhere-vanishing vector field on S^2. However, by the Hairy Ball Theorem, such a vector field cannot exist on S^2. Therefore there is no global section on S^2, and TS^2 is nontrivial.

$\square$

Question 15. *Let M^n be a smooth, closed, oriented manifold of dimension n. Consider the diagonal embedding,*

$$\Delta_M : M \hookrightarrow M \times M \tag{95}$$

$$x \to (x, x) \tag{96}$$

Let ν_{Δ_M} be the normal bundle of this embedding.

1. *Show that there is an isomorphism of vector bundles over M,*

$$\nu_{\Delta_M} \cong TM \tag{97}$$

where TM is the tangent bundle of M.

2. *Let $\tau : M \times M \to T(\nu_{\Delta_M})$ be the Thom collapse map. Here $T(\nu_{\Delta_M})$ is the Thom space of the normal bundle. Consider the composition map in homology,*

$$\phi : H_p(M^n;\mathbb{Z}) \times H_q(M^n;\mathbb{Z}) \xrightarrow{x} H_{p+q}(M \times M;\mathbb{Z}) \xrightarrow{*} H_{p+q}(T(\nu_{\Delta_M});\mathbb{Z}) \xrightarrow{\cap u} H_{p+q-n}(M^n;\mathbb{Z}). \tag{98}$$

The first map in this sequence is the cross product, and the last map in this sequence is the Thom isomorphism in homology, given by capping with the Thom class. Show that this composition map ϕ is equal, up to sign, to the intersection product:

$$\phi(\alpha, \beta) = \pm\alpha \cdot \beta. \tag{99}$$

Proof. 1. We have, for $\Delta(x) = (x, x)$, that $T\Delta(x) \cong \{(x, x, v, v)|x \in M, v \in T_xM\}$. We then calculate the normal bundle ν_{Δ_M}: $\{(v_1, v_2) \in TM \times TM|v \cdot v_1 + v \cdot v_2 = 0\}$. In examining the orthogonality condition, we get that

$$v \cdot v_1 + v \cdot v_2 = 0 \Rightarrow \tag{100}$$

$$v \cdot v_1 = -v \cdot v_2 \tag{101}$$

$$v_1 = -v_2 \tag{102}$$

Therefore the normal bundle is $\{(x, x, v, -v)|x \in M, v \in T_xM\}$. This is isomorphic to TM via the isomorphism

$$\Psi(x, x, v, -v) \mapsto (x, v) \tag{103}$$

$$\Psi^{-1}(x, v) \mapsto (x, x, v, -v) \tag{104}$$

This is injective because, if $(x, x, v, -v) \neq (x', x', v', -v')$ as a set, we get their image under Ψ as (x, v) vs. (x', v'), which are not equivalent. This is surjective because $(x, v) \in TM$ has preimage $(x, x, v, -v)$, which is indeed in ν_{Δ_M}. This is well-defined, because the quotient relations hold under Ψ, as the normal bundle is subject to the same quotient relations as that of TM, and scalar multiplication by -1 doesn't affect anything.

2. For $\alpha \in H_p(M;\mathbb{Z}), \beta \in H_q(M;\mathbb{Z})$, we have their cross product as $\pm[\alpha \times \beta] \in H_{p+q}(M \times M;\mathbb{Z})$. If we take the induced homomorphism in homology of the Thom collapse map of the embedding $MM \times M$, we have that $\pm\tau_*(\alpha \times \beta) \in H_{p+q}(T(\nu_{\Delta_M});\mathbb{Z})$. We know from part a) that $\nu_{\Delta_M} \cong TM$, so $H_{p+q}(T(\nu_{\Delta_M});\mathbb{Z}) \cong H_{p+q}(T(TM);\mathbb{Z})$, so the Thom isomorphism can be applied from $H_{p+q}(T(\nu_{\Delta_M});\mathbb{Z})$ to $H_{p+q-n}(\Delta(M);\mathbb{Z})$, since M is oriented. From the reasoning in Theorem 9.4 in the notes, we have

$$[\pm\tau_*(\alpha \times \beta) \in H_{p+q}(T(\nu_{\Delta_M});\mathbb{Z}) \cong H_{p_q}(T(TM);\mathbb{Z})] \frown [u] \in H^n(T(TM);\mathbb{Z}) \tag{105}$$

$$= \pm[\alpha \times \beta \cap \Delta(M)] \in H_{p+q-n}(\Delta(M);\mathbb{Z}) \cong H_{p+q-n}(M;\mathbb{Z}) \tag{106}$$

Notice that the intersection of $\alpha \times \beta$ with $\Delta(M) \subset M \times M$ are the exact points when α and β intersect: if $(a, b) \in \alpha \times \beta$ is equal to $(x, x) \in \Delta(M) \subset M \times M$, then $x = a \in \alpha, x = b \in \beta$. Thus the class $\alpha \times \beta \cap \Delta(M) \subset M \times M$ is equal to $\pm[\alpha \cap \beta] \in H_{p+q-n}(M;\mathbb{Z})$, which is the intersection product $\pm\alpha \cdot \beta$.

□