Analytic Number Theory Problems

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All mistakes are to be emailed to aszlau@gmail.com. All problems written by Prof. Kannan Soundararajan and Prof. Andrew Glanville's "Multiplicative Number Theory" textbook, unless otherwise marked.

Contents

1 The Prime Number Theorem

1.1 Partial Summation

1.1.1 Different Forms of the Prime Number Theorem

Question 1. *Given the conjecture*

$$\psi(x) := \sum_{n\le x} \Lambda(n) \sim x \tag{1}$$

where

$$\Lambda(n) = \begin{cases} \log p \ \textit{if } n = p^m \textit{ for } p \textit{ prime and } m \ge 1 \\ 0 \ \textit{otherwise} \end{cases} \tag{2}$$

and the conjecture

$$\pi(x) := \sum_{p\le x} 1 \sim \frac{x}{\log x} \tag{3}$$

Use partial summation to prove that (1) and (3) are equivalent and both are equivalent to the conjecture

$$\theta(x) := \sum_{p \le x} \log(p) = x + o(x) \tag{4}$$

Definition 1. ***Partial Summation:*** *Given a sequence* $a_n \in \mathbb{C}$ *and a function* $f : \mathbb{R} \to \mathbb{C}$*, set* $S(t) = \sum_{k \le t} a_k$*, it is easy to conclude that*

$$\sum_{n=A+1}^{B} a_n f(n) = S(B)f(B) - S(A)f(A) - \sum_{n=A}^{B-1} S(n)(f(n+1) - f(n)) \tag{5}$$

and, if f *is continuously differentiable on* $[A, B]$*, then*

$$\sum_{A<n\le B} a_n f(n) = S(B)f(B) - S(A)f(A) - \int_A^B S(t)f'(t)dt \tag{6}$$

Proof. We start with (3). Notice that, if we choose

$$a_n = \begin{cases} 1 \text{ if } n = p \text{ for p prime} \\ 0 \text{ otherwise} \end{cases} \tag{7}$$

and $f(x) = \log x$, then

$$\begin{aligned}
\theta(x) &= \sum_{n\le x} a_n f(n) &(8)\\
&= (\sum_{n\le x} a_n) \log x - \int_2^x (\sum_{n\le t} a_n)(\log t)' dt &(9)\\
&= (\sum_{p\le x} 1) \log x - \int_2^x (\sum_{p\le t} 1)\frac{1}{t} dt &(10)\\
&= \pi(x) \log x - \int_2^x \pi(t)\frac{1}{t} dt &(11)\\
&\sim \frac{x}{\log x} \log x - \int_2^x \frac{t}{t \log t} dt &(12)\\
&\sim x - \int_2^x \frac{1}{\log t} dt &(13)\\
&\sim x - li(x) &(14)
\end{aligned}$$

Thus $\pi(x) \sim \frac{x}{\log x}$ implies $\theta(x) = x + o(x)$

Now we assume $\theta(x) = x + o(x)$. We can easily see that

$$\theta(x) \leq \psi(x) = \sum_{n\leq x} \Lambda(n) = \sum_{p\leq x} \log p \sum_{k\leq \log_p x} 1 = \sum_{p\leq x} \log p [\frac{\log x}{\log p}] \leq \sum_{p\leq x} \log p \frac{\log x}{\log p} \Rightarrow \tag{15}$$

$$x + o(x) \leq \psi(x) \leq \sum_{p\leq x} \log p \frac{\log x}{\log p} \tag{16}$$

Let $f(n) = \frac{\log x}{\log n}$, and let $a_n = \log n$ if n is prime, and 0 otherwise. Then this sum, through partial summation, gives us

$$x + o(x) \leq \psi(x) \leq \theta(x)\frac{\log x}{\log x} - \log 2 \frac{\log x}{\log 2} - \int_2^x \theta(t) \frac{\log x}{t(\log t)^2} dt = x + o(x) \tag{17}$$

Thus $\theta(x) = x + o(x)$ implies $\psi(x) = x + o(x) \sim x$.

Now we assume $\psi(x) \sim x$. As before,

$$\psi(x) \leq \sum_{p\leq x} \log p \frac{\log x}{\log p} = \pi(x) \log x \tag{18}$$

□

1.1.2 Adding reciprocals

Note: my version of the paper has $\sum_{n\leq x}^N \frac{1}{N}$. I'm pretty sure the denominator should be n, as this sum is just 1.

Question 2. *Prove that for any integer $N \geq 1$,*

$$\sum_{n=1}^N \frac{1}{n} = \log N + 1 - \int_1^N \frac{\{t\}}{t^2} dt \tag{19}$$

Deduce that, for any real $x \geq 1$,

$$\sum_{n\leq x} \frac{1}{n} = \log x + \gamma + O(\frac{1}{x}) \tag{20}$$

where γ is the Euler-Mascheroni constant

$$\gamma = \lim_{N\to\infty} (\sum_{n=1}^N \frac{1}{n} - \log N) = 1 - \int_1^\infty \frac{\{t\}}{t^2} dt \tag{21}$$

Note that, for $t \in \mathbb{R}$, $[t]$ is the integral part of t, and $\{t\}$ is the rest of t.

Proof. We use partial summation again. Let $f(x) = \frac{1}{x}$ and $a_n = 1$. Thus, by partial summation, we have

$$\sum_{n\leq x}\frac{1}{n} = [N]\frac{1}{N} + \log 1 + \int_1^N t\frac{1}{t^2}dt \tag{22}$$
$$= [N]\frac{1}{N} + \int_1^N \frac{1}{t^2}(t - \{t\})dt \tag{23}$$
$$= 1 + \int_1^N \frac{t}{t^2}dt - \int_1^N \frac{\{t\}}{t^2}dt \tag{24}$$
$$= 1 + \log N - \log 1 - \int_1^N \frac{\{t\}}{t^2}dt \tag{25}$$
$$= \log N + 1 - \int_1^N \frac{\{t\}}{t^2}dt \tag{26}$$

For any *real* x, we have, through partial summation,

$$\sum_{n\leq x}\frac{1}{n} = [N]\frac{1}{N} + \int_1^N t\frac{1}{t^2}dt \tag{27}$$
$$= \frac{x - \{x\}}{x} + \log N - \int_1^N \frac{\{t\}}{t^2}dt \tag{28}$$
$$= \log N + 1 - \frac{\{x\}}{x} + \int_N^\infty \frac{\{t\}}{t^2}dt - \int_1^\infty \frac{\{t\}}{t^2}dt \tag{29}$$
$$= \log N + \gamma - \frac{\{x\}}{x} + \int_N^\infty \frac{\{t\}}{t^2}dt \tag{30}$$

It remains to prove that $\frac{\{x\}}{x}$ and $\int_N^\infty \frac{\{t\}}{t^2}dt$ are in $O(\frac{1}{x})$. Starting with the former, we see that since $\{x\} < 1$, we have that $|\frac{\{x\}}{x}| < \frac{1}{x}$, so $\frac{\{x\}}{x} \in O(\frac{1}{x})$. Similarly, we have

$$|\int_N^\infty \frac{\{t\}}{t^2}dt| \leq \int_N^\infty |\{t\}||\frac{1}{t^2}|dt \leq \int_N^\infty \frac{1}{t^2}dt \in O(\frac{1}{x}) \tag{31}$$

Thus we conclude

$$\sum_{n\leq x}\frac{1}{n} = \log x + \gamma + O(\frac{1}{x}) \tag{32}$$

□

1.1.3 log$N$!

Question 3. *For an integer $N \geq 1$, show that*

$$\log N! = N\log N - N + 1 + \int_1^N \frac{\{t\}}{t}dt \tag{33}$$

Using that $\int_1^x(\{t\}-1/2)dt = (\{x\}^2-\{x\})/2$*, show that*

$$\int_1^N \frac{\{t\}}{t}dt = \frac{1}{2}\log N - \frac{1}{2}\int_1^N \frac{\{t\}-\{t\}^2}{t^2}dt \tag{34}$$

Conclude that $N! \sim C\sqrt{N}(N/e)^N$*, where you can take as fact that*

$$C = \exp(1 - \frac{1}{2}\int_1^\infty \frac{\{t\}-\{t\}^2}{t^2}dt) = \sqrt{2\pi} \tag{35}$$

Proof. From rules of logarithms, we have $\log N! = \log(N(N-1)...(2)(1)) = \log N + \log(N-1) + ... + \log 2 + \log 1$. We use partial summation once again. Let $a_n = 1$, and $f(x) = \log x$. From the partial summation formula given by (6), we have

$$\log N! = N\log N - 0 - \int_1^N (\sum_{n\leq t} 1)\frac{dt}{t} \tag{36}$$
$$= N\log N - \int_1^N \frac{[t]}{t}dt \tag{37}$$
$$= N\log N - \int_1^N \frac{t-\{t\}}{t}dt \tag{38}$$
$$= N\log N - \int_1^N dt + \int_1^N \frac{\{t\}}{t}dt \tag{39}$$
$$= N\log N - N + 1 + \int_1^N \frac{\{t\}}{t}dt \tag{40}$$

As for the next part, we notice (38):

$$\int_1^N \frac{\{t\}}{t}dt = \int_1^N \frac{\{t\}-\frac{1}{2}+\frac{1}{2}}{t}dt \tag{41}$$
$$= \int_1^N \frac{1}{t}(\{t\}-\frac{1}{2})dt + \int_1^N \frac{1}{2t}dt \tag{42}$$
$$= \frac{1}{t}\frac{\{t\}^2-\{t\}}{2}|_1^N - \int_1^N \frac{1}{2}\frac{\{t\}^2-\{t\}}{-t^2}dt + \frac{1}{2}\log N + \frac{1}{2}\log 1 \tag{43}$$
$$= 0 + \int_1^N \frac{1}{2}\frac{\{t\}^2-\{t\}}{t^2}dt + \frac{1}{2}\log N \tag{44}$$
$$= \frac{1}{2}\log N - \int_1^N \frac{1}{2}\frac{\{t\}-\{t\}^2}{t^2}dt \tag{45}$$

Thus we have

$$\log N! = N\log N - N + 1 + \frac{1}{2}\log N - \frac{1}{2}\int_1^N \frac{\{t\}-\{t\}^2}{t^2}dt \tag{46}$$
$$= N\log N - N + 1 + \frac{1}{2}\log N - \frac{1}{2}\int_1^\infty \frac{\{t\}-\{t\}^2}{t^2}dt + \frac{1}{2}\int_N^\infty \frac{\{t\}-\{t\}^2}{t^2}dt \tag{47}$$
$$= N\log N - N + \frac{1}{2}\log N + \log C + \frac{1}{2}\int_N^\infty \frac{\{t\}-\{t\}^2}{t^2}dt \tag{48}$$

Taking the exponent of both sides, we get

$$N! = N^N \cdot \frac{1}{e^N}\sqrt{N} \cdot C\sqrt{e^{\int_N^\infty \frac{\{t\}-\{t\}^2}{t^2}dt}} \tag{49}$$

Now we examine the integral in the exponent. First we split it up.

$$\frac{1}{2}\int_N^\infty \frac{\{t\}}{t^2}dt - \frac{1}{2}\int_N^\infty \frac{\{t\}^2}{t^2}dt \leq |\frac{1}{2}\int_N^\infty \frac{\{t\}}{t^2}dt| - |\frac{1}{2}\int_N^\infty \frac{\{t\}^2}{t^2}dt| \leq \frac{1}{2}\int_N^\infty |\{t\}||\frac{1}{t^2}|dt - \frac{1}{2}\int_N^\infty |\{t\}^2|\frac{1}{t^2}dt \tag{50}$$

It is easy to see that the limit as N approaches infinity the value of this integral converges to 0, so

$$N! \sim N^N \cdot \frac{1}{e^N}\sqrt{N} \cdot C\sqrt{e^0} \Rightarrow \tag{51}$$

$$N! \sim C\sqrt{N}(N/e)^N \tag{52}$$

□

Definition 2. *The* ***Riemann Zeta Function*** *is given by*

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \prod_p (1 - \frac{1}{p^s})^{-1} \tag{53}$$

1.1.4 The Riemann Zeta Function

Note: My copy of the problem has a later part. In another copy of the book, this part is labeled as a problem which requires complex analysis, something I haven't learned yet, so I'm skipping that for now.

Question 4. *Prove that for Re(s)>1,*

$$\zeta(s) = s\int_1^\infty \frac{[y]}{y^{s+1}}dy = \frac{s}{s-1} - s\int_1^\infty \frac{\{y\}}{y^{s+1}}dy \tag{54}$$

Proof. We use partial summation again. We see that, for $a_n = 1, f(x) = \frac{1}{x^s}$, we have $\zeta(s) =$

$\sum_1^\infty a_n f(n)$, so, using the usual partial summation formula,

$$\zeta(s) = \sum_1^\infty a_n f(n) = \lim_{N\to\infty} \sum_1^N a_n f(n) \tag{55}$$

$$= \lim_{N\to\infty} [[N]\frac{1}{N^s} - [1]\frac{1}{1^s} - \int_1^N [y]\frac{1}{y^s} dy] \tag{56}$$

$$= \lim_{N\to\infty} [[N]\frac{1}{N^s} - [1]\frac{1}{1^s} - [N]\frac{1}{N^s} + [1]\frac{1}{1^s} + s\int_1^N [y]\frac{1}{y^{s+1}} dy] \tag{57}$$

$$= \lim_{N\to\infty} [s\int_1^N [y]\frac{1}{y^s} dy] \tag{58}$$

$$= s\int_1^\infty [y]\frac{1}{y^s} dy \tag{59}$$

We write this final integral in a different way:

$$s\int_1^\infty [y]\frac{1}{y^s} dy = \lim_{N\to\infty} s\int_1^N \frac{y - \{y\}}{y^{s+1}} dy \tag{60}$$

$$= s\int_1^N \frac{y}{y^{s+1}} dy - s\int_1^N \frac{\{y\}}{y^{s+1}} dy \tag{61}$$

$$= \lim_{N\to\infty} [s\int_1^N \frac{1}{y^s} dt - s\int_1^N \frac{\{y\}}{y^{s+1}} dy] \tag{62}$$

$$= \lim_{N\to\infty} [-s\frac{1}{s-1}(\frac{1}{t^{s-1}}|_1^N) - s\int_1^N \frac{\{y\}}{y^{s+1}} dy] \tag{63}$$

$$= \lim_{N\to\infty} [-\frac{s}{s-1}(\frac{1}{N^{s-1}} - \frac{1}{1^{s-1}}) - s\int_1^N \frac{\{y\}}{y^{s+1}} dt] \tag{64}$$

Since $Re(s) > 1$, we have $Re(s) - 1 > 0$, so, evaluating the limit, we get that this expression is equivalent to

$$\frac{-s}{s-1}(0-1) - s\int_1^N \frac{\{y\}}{y^{s+1}} dt = \frac{s}{s-1} - s\int_1^N \frac{\{y\}}{y^{s+1}} dt \tag{65}$$

□

1.2 Chebyshev's Elementary Estimates

1.2.1 $\lim \frac{\psi(x)}{x}$

Question 5. *Prove that*

$$\lim_{x\to\infty} \sup \frac{\psi(x)}{x} \geq 1 \geq \lim_{x\to\infty} \inf \frac{\psi(x)}{x} \tag{66}$$

so that if $\lim_{x\to\infty} \frac{\psi(x)}{x}$ *exists, it must be equal to 1.*

Note that $\log n = \sum_{d|n} \Lambda(d)$, so

$$\sum_{n<x} \log n = \sum_{n\leq x} \sum_{n=dk} \Lambda(d) = \sum_{k=1}^{\infty} \psi(\frac{x}{k}) \tag{67}$$

and, by Stirling's Formula, we have

$$\sum_{k=1}^{\infty} \psi(\frac{x}{k}) = x \log x - x + O(\log x) \tag{68}$$

Proof. We start with $\limsup \frac{\psi(x)}{x} \geq 1$. Suppose not. Then there exists $\epsilon > 0$ such that, for all $x > x_0$ for some $x_0 \geq 2$, $\frac{\psi(x)}{x} \leq (1-\epsilon)$. Then we have

$$\sum_{k=1}^{\infty} \psi(x/k) \leq \sum_{k=1}^{x/x_0} \psi(x/k) + \sum_{x/x_0<k}^{\infty} \psi(x/k) \tag{69}$$

$$x \log x - x + O(\log x) \leq (1-\epsilon)x \sum_{k=1}^{x/x_0} \frac{1}{k} + \sum_{x/x_0<k}^{\infty} \psi(x/k) \tag{70}$$

$$x \log x - x + O(\log x) \leq (1-\epsilon)x(\log x - \log x_0 + \gamma + O(\frac{1}{x})) + \sum_{x/x_0<k}^{\infty} \psi(x/k) \tag{71}$$

$$\epsilon x \log x + O(x) \leq \sum_{x/x_0<k}^{\infty} \psi(x/k) \leq \psi(x_0)(x - x_0) \tag{72}$$

where γ is the Euler-Mascheroni constant. Since the LHS is $O(xlogx)$ and the RHS is $O(x)$, this cannot be true for all $x > x_0$. Thus $\lim_{x\to\infty} \sup \frac{\psi(x)}{x} \geq 1$.

We follow the same approach as with lim sup. We suppose by contradiction that there exists $\epsilon > 0$ such that, for all $x > x_0$ for some $x_0 \geq 2$, such that $\frac{\psi(x)}{x} \geq (1+\epsilon)$. We then have

$$(1+\epsilon)x \sum_{k=1}^{x/x_0} \frac{1}{k} + \sum_{\lfloor x/x_0 \rfloor<k}^{\infty} \psi(x/k) \leq \sum_{k=1}^{\infty} \psi(x/k) \tag{73}$$

$$(1+\epsilon)x[\log \frac{x}{x_0} + \gamma + O(\frac{1}{x})] + \sum_{\lfloor x/x_0 \rfloor<k}^{\infty} \psi(x/k) \leq x \log x - x + O(\log x) \tag{74}$$

$$\frac{1}{x} \sum_{\lfloor x/x_0 \rfloor<k}^{\infty} \psi(x/k) \leq -\epsilon \log x - 1 + (1+\epsilon)\log x_0 - (1+\epsilon)\gamma \tag{75}$$

Since $\psi(x)$ is strictly nonnegative for all $x \in \mathbb{Z}^+$, this cannot be true for all $x > x_0$. Thus

$$\lim_{x\to\infty} \sup \frac{\psi(x)}{x} \geq 1 \geq \lim_{x\to\infty} \inf \frac{\psi(x)}{x} \tag{76}$$

□

1.2.2 Proof of Bertrand's postulate

Question 6. *Given that*

$$\psi(2x) - \psi(x) + \psi(2x/3) \geq x \log 4 + O(\log x)$$

Proof that there exists a prime between N and $2N$ for large N.

It is given to us that $\psi(x) \leq x \log 4 + O((\log x)^2)$. (To see this, just subtract $\psi(2x) - \psi(x)$ using the approximation given in the previous problem). Therefore, we get that $\pi(x) \leq (\log 4 + o(1)) \frac{x}{\log x}$

Proof. First we rearrange terms and take the given bound, resulting in

$$\psi(2x) - \psi(x) \geq x \log 4 - \psi(2x/3) + O(\log x) \Rightarrow \tag{77}$$

$$\psi(2x) - \psi(x) \geq \frac{1}{3} x \log 4 + O(\log x) \tag{78}$$

Notice that $\psi(x) = \sum_{p \leq x} \log p \lfloor \frac{\log x}{\log p} \rfloor$. Then this inequality becomes

$$\sum_{p \leq 2x} \log p \lfloor \frac{\log 2x}{\log p} \rfloor - \sum_{p \leq x} \log p \lfloor \frac{\log x}{\log p} \rfloor \geq \frac{1}{3} x \log 4 + O(\log x) \tag{79}$$

The LHS is less than or equal to

$$\sum_{p \leq 2x} \log 2x - \sum_{p \leq x} \log x \geq \frac{1}{3} x \log 4 + O(\log x)$$

$$[\pi(2x) - \pi(x)] \log x + \pi(2x) \log 2 \geq \frac{1}{3} x \log 4 + O(\log x)$$

$$\pi(2x) - \pi(x) \geq \frac{x}{3 \log x} \log 4 - \pi(2x) \frac{\log 2}{\log x} + O(1)$$

$$\pi(2x) - \pi(x) \geq \frac{2x}{3} \frac{\log 2}{\log x} - \pi(2x) \frac{\log 2}{\log x} + O(1)$$

Thus we have that there exists a prime number between $2x$ and x if $\pi(2x) < \frac{2x}{3}$ for large x. As we bounded $\pi(x) \leq (\log 4 + o(1)) \frac{x}{\log x} < \frac{x}{3}$, this is the case for large enough x. □

1.2.3 $\sum_{n \leq x} \frac{\Lambda(n)}{n}$

Question 7. *It's given that*

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \sum_{p \leq x} \frac{\log p}{p} + O(1) = \log x + O(1)$$

Show that this would follow from the Prime Number Theorem. Why does the Prime Number Theorem not follow from this? What stronger information on $\sum_{p \leq x} \frac{\log p}{p}$ would yield the Prime

Number Theorem?

Proof. The first equality follows from the fact that $\frac{\log x}{x^m}$ <1 for $m > 1$. The second equality follows from partial summation:

$$\begin{aligned}\sum_{1\leq n\leq x} a_n \frac{\log n}{n} &= \pi(x)\frac{\log x}{x} - 0 - \int_1^x \pi(t)\frac{1-\log t}{t^2}dt \\ \text{(PNT)} &\sim 1 - \int_1^x \frac{1}{t\log t}dt - \int_1^x \frac{dt}{t} \\ &= 1 - \log(\log x) + \log x \\ &= \log x + O(1)\end{aligned}$$

□