Math 215C Problems

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Question 1. *Let M be an n-dimensional manifold and g be a metric. Given a $\binom{k}{l}$-tensor field F, define a $\binom{k}{l+1}$-tensor field ∇F by*

$$(\nabla F)(Z, X_1, ..., X_k, \omega_1, ..., \omega_l) := (\nabla_Z F)(X_1, ..., X_k, \omega_1, ..., \omega_l) \tag{1}$$

where we recall

$$\nabla_Z[F(X_1, ..., X_k, \omega_1, ..., \omega_l)] = (\nabla_Z F)(X_1, ..., X_k, \omega_1, ..., \omega_l) + \sum_{i=1}^{k} F(X_1, ..., \nabla_Z X_i, ..., X_k, \omega_1, ..., \omega_l) \tag{2}$$

$$+ \sum_{j=1}^{l} F(X_1, ..., X_k, \omega_1, ..., \nabla_Z \omega_i, ..., \omega_l) \tag{3}$$

where for a function f, $\nabla_Z f = Zf$.

a) Let $f : M \to \mathbb{R}$ be a smooth function. Prove that

$$(\nabla^2 f)(X, Y) = X(Yf) - (\nabla_X Y)f \tag{4}$$

b) Not needed (convince oneself an expression is well-defined).

c) Prove that, in local coordinates,

$$\Delta f := \sum_{i,j=1}^{n} (g^{-1})^{ij} (\nabla^2 f)(\partial_i, \partial_j) = \frac{1}{\sqrt{\det g}} \partial_i((g^{-1})^{ij} \sqrt{\det g} \partial_j f) \tag{5}$$

where one may use the following facts from linear algebra without proof:

$$\partial_i (g^{-1})^{jk} = -(g^{-1})^{jl} (g^{-1})^{mk} \partial_i g_{lm}, \partial_i \log(\det g) = (g^{-1})^{jk} \partial_i g_{jk} \tag{6}$$

Proof. **a**

Expanding $(\nabla^2 f)(X, Y)$, we have this equal to $(\nabla(\nabla f))(X, Y)$ by definition. By the definition of ∇, we have this equal to $(\nabla_X(\nabla f))(Y)$. From class and the footnote, we have

$$\nabla_X[\nabla f(Y)] = (\nabla_X(\nabla f))(Y) + \nabla f(\nabla_X Y) \Rightarrow \tag{7}$$

$$(\nabla_X(\nabla f))(Y) = \nabla_X[\nabla f(Y)] - \nabla f(\nabla_X Y) \tag{8}$$

Again by the definition of ∇, the right-hand side then equal to $\nabla_X[\nabla_Y f] - \nabla_{\nabla_X Y} f$, which, again from class and the footnote, is $\nabla_X[Yf] - (\nabla_X Y)f$. Thinking of $[Yf]$ as another function, we have this equal to $X(Yf) - (\nabla_X Y)f$. Thus $(\nabla^2 f)(X, Y) = X(Yf) - (\nabla_X Y)f$.

c

When we define $\Delta f = \sum_{i,j}^{n}(g^{-1})^{ij}(\nabla^2 f)(\partial_i, \partial_j)$, we can use our formula proved above to obtain $\sum_{i,j}^{n}(g^{-1})^{ij}(\partial_i(\partial_j f) - (\nabla_{\partial_i}\partial_j)f) = \sum_{i,j}^{n}(g^{-1})^{ij}\partial_i(\partial_j f) - (g^{-1})^{ij}(\nabla_{\partial_i}\partial_j)f$. Now we examine the second term, $\sum_{i,j}^{n} -(g^{-1})^{ij}(\nabla_{\partial_i}\partial_j)f$. We will drop the summation in front and use notation with the understanding that repeated indices are summed over. This term is equal, by definition, to $-(g^{-1})^{ij}\Gamma_{ij}^{k}\partial_k f$. We have

$$\begin{aligned} -(g^{-1})^{ij}\Gamma_{ij}^{k}\partial_k f &= -(g^{-1})^{ij}(\frac{1}{2}(g^{-1})^{kl}(\partial_i g_{j,l} + \partial_j g_{il} - \partial_l g_{ij}))\partial_k f && (9)\\ &= \frac{1}{2}(-(g^{-1})^{ij}(g^{-1})^{kl}\partial_i g_{jl} - (g^{-1})^{ij}(g^{-1})^{kl}\partial_j g_{il} + (g^{-1})^{ij}(g^{-1})^{kl}\partial_l g_{ij})\partial_k f && (10)\\ &= (\frac{-1}{2}(-g^{-1})^{ij}(g^{-1})^{kl}\partial_i g_{jl} + \frac{-1}{2}(g^{-1})^{ij}(g^{-1})^{kl}\partial_j g_{il})\partial_k f + \frac{1}{2}((g^{-1})^{ij}(g^{-1})^{kl}\partial_l g_{ij})\partial_k f && (11) \end{aligned}$$

In examining the first two terms, since the indices are dummy indices, we can relabel indices and rewrite their sum as $-(g^{-1})^{ij}(g^{-1})^{kl}(\partial_i g_{jl})\partial_k f$. We know from the first hint that this is equal to $(\partial_i(g^{-1})^{jk})\partial_k f$, so we now have

$$\begin{aligned} &= (\partial_i(g^{-1})^{jk})\partial_k f + \frac{1}{2}(g^{-1})^{ij}(g^{-1})^{kl}\partial_l g_{ij}\partial_k f && (12)\\ &= (\partial_i(g^{-1})^{jk})\partial_k f + \frac{1}{2}(\partial_l \log(\det g))\partial_k f && (13)\\ & && (14) \end{aligned}$$

as per the second hint. Then,

$$\begin{aligned} &= (\partial_i(g^{-1})^{jk})\partial_k f + (\partial_l \log(\sqrt{\det g}))\partial_k f && (15)\\ &= (\partial_i(g^{-1})^{jk})\partial_k f + \frac{1}{\sqrt{\det g}}(\partial_l \sqrt{\det g})\partial_k f && (16)\\ & && (17) \end{aligned}$$

Putting this all together, we have

$$\Delta f = (g^{-1})^{ij}\partial_i(\partial_j f) + (\partial_i(g^{-1})^{jk})\partial_k f + \frac{1}{\sqrt{\det g}}(g^{-1})^{ij}(\partial_l \sqrt{\det g})\partial_k f \tag{18}$$

We now show that this is equal to $\frac{1}{\sqrt{\det g}}\partial_i((g^{-1})^{ij}\sqrt{\det g}\partial_i f)$. Since ∂_i is a derivation, we can use the 'product rule' and apply ∂_i to each of the three terms:

$$\begin{aligned} \frac{1}{\sqrt{\det g}}\partial_i((g^{-1})^{ij}\sqrt{\det g}\partial_j f) &= \frac{1}{\sqrt{\det g}}((\partial_i(g^{-1})^{ij})\sqrt{\det g}\partial_j f + (g^{-1})^{ij}(\partial_i\sqrt{\det g})\partial_j f + (g^{-1})^{ij}\sqrt{\det g}\partial_i\partial_j f) && (19)\\ &= (\partial_i(g^{-1})^{ij})\partial_j f + \frac{1}{\sqrt{\det g}}(g^{-1})^{ij}(\partial_i\sqrt{\det g})\partial_j f + (g^{-1})^{ij}\partial_i(\partial_j f) && (20) \end{aligned}$$

By relabeling our dummy indices, we see that this is equal to (5). □

Question 2. *Let* $\gamma : (a, b) \to \mathbb{R}^3$ *be a smooth curve defined by* $\gamma(s) = (x(s), 0, z(s))$ *where* $(x'(s))^2 + (z'(s))^2 = 1$ *and* $x(s) > 0$ *for all* $s \in (a, b)$*. Let* M *denote the surface of revolution obtained by rotating* γ *around the* z *axis:*

$$M = \{(x(s)\cos\theta, x(s)\sin\theta, z(s)) | s \in (a, b), \theta \in S^1\} \tag{21}$$

and let $g = ds^2 + x(s)^2 d\theta^2$ *denote the induced metric from* $(\mathbb{R}^3, g_{Eucl})$*.*

1. *Compute the Christoffel symbols* Γ^k_{ij} *of the Levi-Civita connection of* g *with respect to the frame* $\{\partial_s, \partial_\theta\}$*.*
2. *Write down the geodesic equations of a curve* $c(t) := (s(t), \theta(t))$ *in* M
3. *Let* $c(t)$ *be such that* $|c'(t)|_g = 1$ *for all* t*, and let* $\phi(t)$ *denote the angle between* $c'(t)$ *and* ∂_θ*. Show that* $x(s(t))\cos\phi(t)$ *is independent of* t*.*

Proof. **a**

By the definition of our Christoffel symbols, we have

$$\Gamma^k_{ij} = \frac{1}{2}(g^{-1})^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \tag{22}$$

Given our metric $g = ds^2 + x(s)^2 d\theta^2$, we see that $g_{ss} = 1, g_{s\theta} = g_{\theta s} = 0, g_{\theta\theta} = x(s)^2$. As a result, $(g^{-1})^{ss} = 1, (g^{-1})^{s\theta} = (g^{-1})^{\theta s} = 0, (g^{-1})^{\theta\theta}) = \frac{1}{x(s)}$ Thus we have

$$\Gamma^s_{ss} = \frac{1}{2}(g^{-1})^{ss}(2\partial_s g_{ss} - \partial_s g_{ss}) = \frac{1}{2} * 1 * (2\partial_s(1) - \partial_s(1)) = 0 \tag{23}$$

$$\Gamma^\theta_{ss} = \frac{1}{2}(g^{-1})^{\theta\theta}(0 + 0 - \partial_\theta g_{ss}) = 0 \tag{24}$$

$$\Gamma^s_{s\theta} = \Gamma^s_{\theta s} = \frac{1}{2}(g^{-1})^{ss}(\partial_s g_{\theta s} + \partial_\theta g_{ss}) = \frac{1}{2}(\partial_s(0) + \partial_\theta(1)) = 0 \tag{25}$$

$$\Gamma^\theta_{s\theta} = \Gamma^\theta_{\theta s} = \frac{1}{2}(g^{-1})^{\theta\theta}(\partial_s g_{\theta\theta} + \partial_\theta g_{s\theta} = \frac{1}{2x(s)^2}(\partial_s x(s)^2 + \partial_\theta(0)) = \frac{x'(s)}{x(s)} \tag{26}$$

$$\Gamma^s_{\theta\theta} = \frac{1}{2}(\partial_\theta g_{\theta s} + \partial_\theta g_{\theta s} - \partial_s g_{\theta\theta}) = \frac{1}{2}(\partial_\theta(0) + \partial_\theta(0) - \partial_s(x(s)^2)) = \frac{1}{2}(0 + 0 + 2x(s)x'(s)) = -x(s)x'(s) \tag{27}$$

$$\Gamma^\theta_{\theta\theta} = \frac{1}{2x(s)^2}(\partial_\theta g_{\theta\theta}) = \frac{1}{2x(s)^2}(\partial_\theta(x(s)^2)) = 0 \tag{28}$$

b

A geodesic satisfies the equation $D_t(c'(t)) = 0$ i.e. $\frac{d^2c(t)}{dt^2} + \Gamma^i_{jk}(\frac{dc(t)}{dt})^j(\frac{dc(t)}{dt})^k = 0$. Thus we have the equations: (where 'stuff that vanishes' denotes factors of Christoffel symbols that are zero)

$$\frac{d^2}{dt^2}s(t) = -\Gamma^s_{ss}(\frac{ds(t)}{dt})^2 - \Gamma^s_{\theta s}\frac{d\theta(t)}{dt}\frac{ds(t)}{dt} - \Gamma^s_{s\theta}\frac{ds(t)}{dt}\frac{d\theta(t)}{dt} - \Gamma^s_{\theta\theta}(\frac{d\theta(t)}{dt})^2 \tag{29}$$

$$= x(s)x'(s)(\frac{d\theta(t)}{dt})^2, \tag{30}$$

$$\frac{d^2}{dt^2}\theta(t) = -\Gamma^\theta_{ss}(\text{stuff that vanishes}) - 2\Gamma^\theta_{\theta s}(\frac{ds(t)}{dt})(\frac{d\theta(t)}{dt}) - \Gamma^\theta_{\theta\theta}(\text{stuff that vanishes}) \tag{31}$$

$$= -2\frac{x'(s(t))}{x(s(t))}\frac{ds(t)}{dt}\frac{d\theta(t)}{dt} \tag{32}$$

These are our geodesic equations.

c

$\cos(\phi(t)) = \frac{g(c'(t),\partial_\theta)}{|c'(t)||\partial_\theta|}$ by definition. We are given that $|c'(t)| = 1$, and $|\partial_\theta| = \sqrt{g(\partial_\theta, \partial_\theta)} = \sqrt{x(s)^2} = x(s)$. This is well-defined because, for all s in our range, $x(s) > 0$. Thus $x(s)\cos(\phi(t)) = g(c'(t), \partial_\theta)$. Due to linearity of g, $g(c'(t), \partial_\theta) = g(s'(t) + \theta'(t), \partial_\theta) = g(s'(t), \partial_\theta) + g(\theta'(t), \partial_\theta) = 0 + x(s)^2\frac{d\theta(t)}{dt}$. It remains to show that this is constant with respect to t. We can do this by showing the derivative of this expression with respect to t is 0. Differentiating with respect to t, we use the 'product rule' of our derivation to find that

$$\frac{d(x(s)\cos(\phi(t))}{dt} = \frac{d(x(s)^2\frac{d\theta(t)}{dt})}{dt} = 2x(s)x'(s)\theta'(t) + x(s)^2\theta''(t) \tag{33}$$

With knowledge that $c(t)$ is our geodesic, we substitute in our equation for $\theta''(t)$:

$$= 2x(s(t))x'(s(t))s'(t)\theta'(t) - 2x(s(t))^2s'(t)\frac{x'(s(t))}{x(s(t))}s'(t)\theta'(t) \tag{34}$$

$$= 2x(s(t))x'(s(t))s'(t)\theta'(t) - 2x(s(t))x'(s(t))s'(t)\theta'(t) = 0 \tag{35}$$

Thus $x(s(t))^2\cos(\phi(t))$ does not depend on t. □

Question 3. *Let $\mathbb{S}^2$ denote the standard 2-sphere embedded in $\mathbb{R}^3$. Let $U \subset \mathbb{S}^2$ be the open subset defined by $U = \{(x^1, x^2, x^3) \in \mathbb{S}^2 : \frac{1}{2} < x^3 < \frac{1}{2}\}$.*

1. *Let $V \subseteq U$ be an open set. Are there nonzero vector fields Y defined in V such that $\nabla_X Y = 0$ for all X?*
2. *Using part a), conclude that $\mathbb{S}^2$ is not locally isometric to $\mathbb{R}^2$.*

Proof. **a**

We can parametrize U inspired by the previous question:

$$U = \{(x(s)\cos(\theta), x(s)\sin(\theta), z(s))|s \in (-\frac{\pi}{6}, \frac{\pi}{6}), \theta \in S^1, x(s) = \cos(s), z(s) = \sin(s)\} \tag{36}$$

We can use the metric of the previous problem because $\cos(s) > 0$ for our s and $(x'(s))^2 + (z'(s))^2 = (-\sin(s))^2 + \cos^2(s) = 1$. In finding $\nabla_X Y$, we use the Christoffel symbols found in the previous problem, plugging in $x(s) = \cos(s)$. We see to find a $Y := y_s\partial_s + y_\theta\partial_\theta$ such that $\nabla_X Y = 0$ for all $X := x_s\partial_s + x_\theta\partial_\theta$ in V.

$$\nabla_X Y = \sum_k (\sum_{i,j} x_i y_j \Gamma^k_{ij} + X(y_k))X_k \tag{37}$$

$$= (x_s y_s \Gamma^s_{ss} + x_s y_\theta \Gamma^s_{s\theta} + x_\theta y_s \Gamma^s_{\theta s} + x_\theta y_\theta \Gamma^s_{\theta\theta} + x_s \partial_s y_s + x_\theta \partial_\theta y_s) x_s \partial_s \tag{38}$$

$$+ (x_s y_s \Gamma^\theta_{ss} + x_s y_\theta \Gamma^\theta_{s\theta} + x_\theta y_s \Gamma^\theta_{\theta s} + x_\theta y_\theta \Gamma^\theta_{\theta\theta} + x_s \partial_s y_\theta + x_\theta \partial_\theta y_\theta) x_\theta \partial_\theta \tag{39}$$

$$= (0 + 0 + 0 + x_\theta y_\theta x'(s)x(s) + x_s \partial_s y_s + x_\theta \partial_\theta y_s) x_s \partial_s \tag{40}$$

$$+ (0 + x_s y_\theta \frac{x'(s)}{x(s)} + x_\theta y_s \frac{x'(s)}{x(s)} + 0 + x_s \partial_s y_\theta + x_\theta \partial_\theta y_\theta) x_\theta \partial_\theta \tag{41}$$

Thus it remains to solve

$$x_\theta y_\theta x'(s)x(s) + x_\theta \partial_\theta y_s + x_s \partial_s y_s = 0, \tag{42}$$

$$x_s y_\theta \frac{x'(s)}{x(s)} + x_\theta y_s \frac{x'(s)}{x(s)} + x_s \partial_s y_\theta + x_\theta \partial_\theta y_\theta = 0 \tag{43}$$

Notice that in V, $x(s)$ is never 0. Suppose now that $s \neq 0$, because if we find a Y that satisfies these two equations in $s \neq 0$, by smoothness of vector fields it must satisfy this equation for $s = 0$ too. Suppose X is such that $x_\theta = 0, x_s \neq 0$. Then our top equation reduces to $\partial_s y_s = 0$, so y_s is constant with respect to s since we assumed $x_s \neq 0$. Now suppose X is such that $x_s = 0, x_\theta \neq 0$. Then we know $y_s \tan(s) = \partial_\theta y_\theta$. Since we are assuming $s \neq 0$ for now, we have $y_s = \cot(s)\partial_\theta y_\theta$. But since we know that y_s must be constant with respect to s, we must have $\partial_\theta y_\theta = 0$, which means y_θ is constant with respect to θ, and $y_s = 0$. Plugging in $y_s = 0$ to (35), we get that $-x_\theta y_\theta \sin(s)\cos(s) = 0$ which implies that $y_\theta = 0$. Thus, the only Y on V that yields $\nabla_X Y$ for any and all X on V is $Y = 0$, assuming $s \neq 0$. But to preserve smoothness, Y must also be zero on $s = 0$ as well. There there does not exist a nonzero vector field Y on V such that $\nabla_X Y = 0$ for all X.

b

A local isometry preserves the metric, and thus preserves our connection, as we used our connection in terms of our Christoffel symbols, which in problem 2 we defined in terms of our metric:

$$\nabla_X Y = \sum_k (\sum_{i,j} x_i y_j \Gamma^k_{ij} + X(y_k))X_k \tag{44}$$

$$= \sum_k (\sum_{i,j} x_i y_j (\frac{1}{2}(g^{-1})^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})) + X(y_k))X_k \tag{45}$$

Thus if we preserve our metric we preserve this connection. Choose $p \in U$ defined in part a). For any open neighborhood W of p, choose a neighborhood V' around p small enough that $V' \subseteq V$, where V was the neighborhood used in part a), so that there exist no nonzero vector fields Y in V' with $\nabla_X Y = 0$ for all X. In a neighborhood of $\mathbb{R}^2$, there exists a nonzero vector field Y such that $\nabla_X Y = 0$ for all X: let $Y = Y^i \frac{\partial}{\partial x^i}, Y^i \in \mathbb{R}$. In our flat space $\mathbb{R}^2$, we choose cartesian coordinates so $g_{ij} = \delta_{ij}$, and so $\partial_i g_{kj} = 0$ for $i, j, k \in \{1, 2\}$, thus all of our Christoffel symbols vanish. Indeed, we have $\nabla_X Y := \nabla_X (Y_i \frac{\partial}{\partial x^i}) = X_j \frac{\partial Y^i}{\partial x^j} \frac{\partial}{\partial x_i} + X^j Y^i \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}$. Since Y^i is independent of x^j, $\frac{\partial Y^i}{\partial x^j} = 0$ and so the first summand is equal to zero. The second summand is also equal to zero because all Christoffel symbols in our coordinate choise are zero. But since there is clearly, in any arbitrary neighborhood of $\mathbb{R}^2$ a nonzero Y such that $\nabla_X Y = 0$ for all X, ∇ cannot be preserved with our isometry because there is no corresponding Y in $V' \subset V$ that has this property. Thus S^2 is not locally isometric to $\mathbb{R}^2$. □

Question 4. *Let (M, g) be a Riemannian manifold. Let $p \in M$ and W be a totally normal neighborhood of p. For every $\epsilon > 0$ such that $B(p, 3\epsilon) \subset W$, define*

$$W_\epsilon := \{(q, V, t) \in TM \times \mathbb{R} : q \in B(q, \epsilon), V \in T_q M, |V| = 1, |t| < 2\epsilon\} \tag{46}$$

and a smooth function $F : W_\epsilon \to \mathbb{R}$ by

$$F(q, V, t) = [d(\exp_q(tV), p)]^2 \tag{47}$$

1. *After choosing $\epsilon > 0$ sufficiently small, prove that $\frac{\partial^2 F}{\partial t^2} > 0$ on W_ϵ.*
2. *Let ϵ be as in a). If $q_1, q_2 \in B(p, \epsilon)$ and γ is a minimizing geodesic from q_1 to q_2, show that $d(\gamma(t), p)$ attains its maximum at one of the endpoints of γ.*
3. *Prove that $B(q, \epsilon)$ is convex.*

Proof. **a**

Let F be defined as in the problem statement. Since our domain of F is $(q, t) \in B(p, \epsilon) \times \mathbb{R} \subseteq W \times \mathbb{R}$, a totally normal neighborhood, the exponential map is well-defined. Since geodesics are locally length-minimizing curves, choose ϵ small enough such that there exists a minimizing geodesic from p to q for any q in our $B(p, 2\epsilon)$ domain. This means that the distance between two points is the length of the geodesic with those two points as endpoints. We then have

$$F(q, V, t) = [d(\exp_q(tV), p)]^2 \tag{48}$$

$$= [d(\gamma(1, q, tV), p)]^2 \tag{49}$$

We know from class that $\gamma(1, q, tV) = \gamma(t, q, V)$. Following the hint, we plug in $p = q$:

$$= [d(\gamma(t,p,V),p)]^2 \tag{50}$$

$$= [\int_0^t |\dot\gamma(t',p,V)|dt']^2 \tag{51}$$

$$= [\int_0^t \sqrt{g(\dot\gamma(t',p,V),\dot\gamma(t',p,V))}dt']^2 \tag{52}$$

since p is $\gamma(0,p,V)$ for the γ provided by the definition of the exponential map. From now on we will simplify notation and denote $\dot\gamma(t',p,V)$ by $\dot\gamma(t')$. Differentiating (5) by t, we have:

$$\frac{d}{dt}([\int_0^t \sqrt{g(\dot\gamma(t'),\dot\gamma(t'))}dt']^2) = 2[\int_0^t \sqrt{g(\dot\gamma(t'),\dot\gamma(t'))}dt'] \cdot \frac{d}{dt}(\int_0^t \sqrt{g(\dot\gamma(t'),\dot\gamma(t'))}dt') \tag{53}$$

For the second factor, we have by the Leibniz Integral Rule,

$$\frac{d}{dt}(\int_0^t \sqrt{g(\dot\gamma(t'),\dot\gamma(t'))}dt') = \sqrt{g(\dot\gamma(t),\dot\gamma(t))}\frac{d}{dt}(t) - \sqrt{g(\dot\gamma(0),\dot\gamma(0))}\frac{d}{dt}(0) + \int_0^t \frac{d}{dt}\sqrt{g(\dot\gamma(t'),\dot\gamma(t'))}dt' \tag{54}$$

$$= |\dot\gamma(t,q,V)| - 0 + \int_0^t [\frac{g(\frac{D}{dt}(\dot\gamma(t')),\dot\gamma(t')) + g(\dot\gamma(t'),\frac{D}{dt}(\dot\gamma(t'))]dt'}{2\sqrt{g(\dot\gamma(t),\dot\gamma(t))}} \tag{55}$$

$$= |\dot\gamma(t,q,V)| - 0 + 0 \tag{56}$$

since 0 is a constant, and γ is a geodesic, so by definition $\frac{D}{dt}\dot\gamma = 0$. Thus

$$\frac{dF}{dt} = 2[\int_0^t \sqrt{g(\dot\gamma(t'),\dot\gamma(t'))}dt'] \cdot \sqrt{g(\dot\gamma(t),\dot\gamma(t))} \tag{57}$$

Differentiating again, using (8) and (9), we get

$$\frac{d^2F}{dt^2} = 2g(\dot\gamma(t),\dot\gamma(t)) + 2[\int_0^t \sqrt{g(\dot\gamma(t'),\dot\gamma(t'))}dt'] \cdot (\frac{g(\frac{D}{dt}(\dot\gamma(t')),\dot\gamma(t')) + g(\dot\gamma(t'),\frac{D}{dt}(\dot\gamma(t')))}{2\sqrt{g(\dot\gamma(t'),\dot\gamma(t')}}) \tag{58}$$

$$= 2g(\dot\gamma(t),\dot\gamma(t)) + 0 \tag{59}$$

We have that $\frac{d^2F}{dt^2}(p,V,0) = 2|V|^2 = 2$. Since F is smooth, this second derivative is continuous, so we can choose ϵ small enough such that for $t < 2\epsilon$ within $B(q,\epsilon) \subseteq M$ we have $\frac{d^2F}{dt^2} > 0$.

b

First we examine $[d(\gamma(t),p)]^2$. Because we are in the same domain as above, we can create a geodesic between any two points in $B(p,\epsilon)$. The distance between any two points in $B(p,\epsilon)$ has to be less than 2ϵ, and so we can have $\exp_{q_1}(tV)$ be any point in $B(p,\epsilon)$ for any $q_1 \in B(p,\epsilon)$ with $|V| = 1$, depending on V. Because we are in a totally normal neighborhood of p, exp is a diffeomorphism, and thus we can choose V

such that $|V| = 1$ and $\exp_q(t_2V) = q_2$ for some $t_2 < 2\epsilon$ ($|t_2V|$ is whatever length we need to get to q_2; we are all good because $|t_2V| \leq |t_2||V| < 2\epsilon$). By uniqueness of geodesics, this map coincides with γ from q_1 to q_2. Thus we can consider $F(q_1, V, t_2)$ instead of $[d(\gamma(t), p)]^2$.

Since ϵ is given as before, we are assured that $F(q_1, V, t)$ is concave up everywhere for $0 \leq t \leq t_2$. This means there cannot be a $t' \in (0, t_2)$ where $F(q_1, V, t')$ attains a maximum. If there where, F would have to be concave down at t', violating what we proved in a). Thus the maximum value of F is achieved at the boundary, either at $t = 0$ or $t = t_2$, i.e.

$$\max[F(q_1, V, t)] = \max[[d(\exp_{q_1}(tV), p)]^2] = \max[[d(\gamma(t), p)]^2] \tag{60}$$

$$= \max[[d(\gamma(0), p)]^2, [d(\gamma(t_2), p)]^2] \tag{61}$$

$$= \max[[d(q_1, p)]^2, [d(q_2, p)]^2] \tag{62}$$

where (13) is by the uniqueness of geodesics, (14) is due to part a), and (15) is by definition.

If $a^2 > b^2, a, b \in \mathbb{R}^{+,0}$, then $a > b$. We know that distance nonnegative. Thus, since the maximum of the distance squared is achieved at the endpoints of γ, the maximum of the distance is achieved at the endpoints of γ.

c

Suppose we have arbitrary $q_1, q_2 \in B(p, \epsilon)$, and the geodesic between the two. Since $q_1, q_2 \in B(p, \epsilon)$, $d(q_1, p) < \epsilon$ and $d(q_2, p) < \epsilon$. Since we proved in b) that the maximum distance between $\gamma(t)$ and p is attained on the endpoints of $\gamma(t)$, i.e. q_1 or q_2. Thus, $d(\gamma(t), p) < \epsilon$ for all t, since the maximum distance, either $d(q_1, p)$ or $d(q_2, p)$, is still less than ϵ. Thus $\gamma(t)$ is completely contained in $B(q, \epsilon)$. Since q_1, q_2 were arbitrary, this works for every $q_1, q_2 \in B(p, \epsilon)$, so $B(p, \epsilon)$ is convex. □

Question 5. *Give an example of a Riemannian manifold that is not complete, but so that every p and q can be joined by a length-minimizing geodesic. Give an example of a Riemannian manifold for which any two points p and q can be joined by a geodesic, but for which there exist two points $\tilde{p}\tilde{q}$ which cannot be joined by a length-minimizing geodesic.*

Proof. The plane $\{(x, y) \in \mathbb{R}^2; y > 0\}$ with metric $g_{11} = g_{22} = \frac{1}{y^2}, g_{12} = g_{21} = 0$ is not complete (consider the sequence $a_n = (x_0, \frac{1}{n})$. Trivially the sequence's limit point is $(x_0, 0), x_0 \in \mathbb{R}$, which is not in the upper half plane), and we know from Do Carmo p.g. 73 that, for each pair of points in the upper half-plane, there is a length minimizing geodesic between the two.

As for a manifold where any points p and q can be joined by a geodesic, but there exist $\tilde{p}, \tilde{q}$ can cannot be joined by a length-minimizing geodesic, consider a punctured sphere: $M = S^2 - \{0, 0, 1\}$. Any two points in M can be joined by a great circle as shown in class. However, let $\tilde{p} = (\frac{1}{2}, 0, \frac{\sqrt{3}}{2}), \tilde{q} = (-\frac{1}{2}, 0, \frac{\sqrt{3}}{2})$. The length-minimizing geodesic would be the arc parametrized by $\gamma(t) = (\cos(t\frac{\pi}{3} + \frac{\pi}{3}), 0, \sin(t\frac{\pi}{3} + \frac{\pi}{3}))$, but this passes through $(0, 0, 1)$. However, the arc parametrized by $(\cos(-t\frac{2\pi}{3} + \frac{\pi}{3}), 0, \sin(-t\frac{2\pi}{3} + \frac{\pi}{3}))$ is a geodesic with $\tilde{p}, \tilde{q}$ as endpoints. □

Question 6. *Let (M, g) be a Riemannian manifold. Suppose that (M, g) is locally conformally flat. Show that*

$$\kappa_{ij} = \frac{1}{\rho^2}((\partial_i \log \rho)^2 + (\partial_j \log \rho)^2 - \sum_{l=1}^{n}(\partial_l \log \rho)^2 - \partial_i^2 \log \rho - \partial_j^2 \rho) \tag{63}$$

where, for each $p \in U, \kappa_{ij}(p) := \kappa(\prod_{ij,p})$ is the sectional curvature of the 2-plane $\prod_{ij,p} \subset T_pM$ spanned by ∂_i and ∂_j.

Proof. First we calculate the Christoffel symbols with this metric. We have

$$\begin{aligned}
\Gamma_{ij}^k &= \frac{1}{2}\sum_l g^{lk}(\partial_i g_{il} + \partial_j g_{li} - \partial_l g_{ij}) && (64)\\
&= \frac{1}{2}\sum_l \frac{1}{\rho^2}\delta_{lk}(\partial_i(\rho^2\delta_{jl}) + \partial_j(\rho^2\delta_{li}) - \partial_l(\rho^2\delta_{ij})) && (65)\\
&= \frac{1}{2}\sum_l \frac{1}{\rho^2}\delta_{lk}(\delta_{jl}2\rho\partial_i\rho + \rho^2\partial_i\delta_{jl} + \delta_{li}2\rho\partial_j\rho + \rho^2\partial_j\delta_{li} - \delta_{ij}2\rho\partial_l\rho - \rho^2\partial_l\delta_{ij}) && (66)\\
&= \frac{1}{2}\sum_l \frac{1}{\rho^2}\delta_{lk}(\delta_{jl}2\rho\partial_i\rho + \delta_{li}2\rho\partial_j\rho - \delta_{ij}2\rho\partial_l\rho) && (67)\\
&= \sum_l(\frac{1}{\rho}\delta_{lk}\delta_{jl}(\partial_i\rho) + \frac{1}{\rho}\delta_{lk}\delta_{li}(\partial_j\rho) - \frac{1}{\rho}\delta_{ij}\delta_{lk}(\partial_l\rho)) && (68)\\
&= \delta_j^k\frac{1}{\rho}(\partial_i\rho) + \delta_i^k\frac{1}{\rho}(\partial_j\rho) - \delta_{ij}\delta^{lk}\frac{1}{\rho}(\partial_l\rho) && (69)\\
&= \delta_j^k\partial_i(\log\rho) + \delta_i^k\partial_j(\log\rho) - \delta_{ij}\delta^{lk}\partial_l(\log\rho) && (70)
\end{aligned}$$

Where (22) is via the definition of the metric, (23) is due to the definition of a derivation, (24) is due to δ being scalar-valued, (26) is splicing the deltas together, and (27) is using the axiom of a derivation.

Next we compute R_{ijij}. In keeping consistent with notation in Do Carmo, we will have $X_i := \partial_i$. We have

$$\begin{aligned}
R_{ijij} &= \langle R(X_i, X_j)X_i, X_j\rangle && (71)\\
&= R_{iji}^l g_{li} && (72)\\
&= R_{iji}^l \rho^2\delta_{li} && (73)\\
&= R_{iji}^j \rho^2 && (74)\\
&= (\sum_l \Gamma_{ii}^l\Gamma_{jl}^j - \sum_l \Gamma_{ji}^l\Gamma_{il}^j + \partial_j\Gamma_{ii}^j - \partial_i\Gamma_{ji}^j)\rho^2 && (75)
\end{aligned}$$

Calculating our Christoffel symbols, we have

$$\Gamma_{ii}^l = \delta_i^l \partial_i \log\rho + \delta_i^l \partial_i \log\rho - \partial_l \log\rho \tag{76}$$

$$\Gamma_{jl}^j = \delta_l^j \partial_j \log\rho + \partial_l \log\rho - \delta_{jl} \partial_j \log\rho \tag{77}$$

$$\Gamma_{ji}^l = \delta_i^l \partial_j \log\rho + \delta_j^l \partial_i \log\rho \tag{78}$$

$$\Gamma_{il}^j = \delta_l^j \partial_i \log\rho - \delta_{il} \partial_j \log\rho \tag{79}$$

$$\Gamma_{ii}^j = -\partial_j \log\rho \tag{80}$$

$$\Gamma_{ji}^j = \partial_i \log\rho \tag{81}$$

Since now i, j are fixed and distinct. In sparing the grader many lines of boring computation, we shall provide intermediate solutions:

$$\sum_l \Gamma_{ii}^l \Gamma_{jl}^j = 2(\partial_i \log\rho)^2 - \sum_l (\partial_l \log\rho) \tag{82}$$

$$\sum_l \Gamma_{ji}^l \Gamma_{il}^j = -(\partial_j \log\rho)^2 + (\partial_i \log\rho)^2 \tag{83}$$

and, trivially,

$$\partial_j \Gamma_{ii}^j = -\partial_j^2 \log\rho \tag{84}$$

$$\partial_i \Gamma_{ji}^j = \partial_i^2 \log\rho \tag{85}$$

Thus $R_{ijij} = (\partial_i \log\rho)^2 + (\partial_j \log\rho)^2 - \sum_l (\partial_l \log\rho)^2 - \partial_j^2 \log\rho - \partial_i^2 \log\rho$. Since $\kappa_{ij} = \frac{R_{ijij}}{|X_i \wedge X_j|^2}$, it remains to calculate the denominator.

$$|X_i \wedge X_j|^2 = (\sqrt{|X_i|^2 |X_j|^2 - g(X_i, X_j)^2})^2 \tag{86}$$

$$= (\sqrt{(\sqrt{\rho^2})^2 (\sqrt{\rho^2})^2 - 0})^2 \tag{87}$$

$$= \rho^2 \tag{88}$$

Thus,

$$\kappa_{ij} = \frac{(\partial_i \log\rho)^2 + (\partial_j \log\rho)^2 - \sum_l (\partial_l \log\rho)^2 - \partial_j^2 \log\rho - \partial_i^2 \log\rho}{\rho^2} \tag{89}$$

□

Question 7. *Let* $\overline{M} = \mathbb{R}^{n+1}$ *with the metric* $\overline{g} = (dx^1)^2 + ... + (dx^n)^2 - (dx^{n+1})^2$. *Define* $M = \{(x^1, ..., x^n, x^{n+1}) \in \mathbb{R}^{n+1} : (x^{n+1})^2 - \sum_{i=1}^n (x^i)^2 = 1\}$ *and suppose* g *is the induced metric. Show that* (M, g) *has constant sectional curvature -1.*

Proof. First we find out induced metric. We examine the tangent vectors of M. Consider the tangent vector ∂_i of $\mathbb{R}^{n+1}$. In visualizing the tangent vector, we see for the i^{th} tangent vector, we see a contribution from the ∂_i vector and some contribution of the ∂_{n+1} vector. Thus we write this tangent vector as $\partial_i + h^i\partial_{n+1}$ for some $h^i \in \mathbb{R}$. M is given by $\sum^n (x^i)^2 - (x^{n+1})^2 + 1 = 0$, so we have

$$0 = (\partial_i + h^i\partial_{n+1})(\sum^n (x^i)^2 - (x^{n+1})^2 + 1) \tag{90}$$

$$= 2x^i - 2h^i x^{n+1} \tag{91}$$

$$\Rightarrow h^i = \frac{x^i}{x^{n+1}} \tag{92}$$

So our tangent vectors are $h^i = \frac{x^i}{x^{n+1}}$. For $i = 1, 2, ..., n$, these span the tangent vectors of M. Thus our metric is $g_{ij} = g(\partial_i + \frac{x^i}{x^{n+1}}\partial_{n+1}, \partial_j + \frac{x^j}{x^{n+1}}\partial_{n+1}) = \overline{g}(\partial_i + \frac{x^i}{x^{n+1}}\partial_{n+1}, \partial_j + \frac{x^j}{x^{n+1}}\partial_{n+1})$. Since $\overline{g}$ is diagonal, this result is $\delta_{ij} - \frac{x^i x^j}{(x^{n+1})^2}$.

We now check that the g given above is Riemannian, because $\overline{g}$ is not Riemannian. Consider

$$g(\sum_i^n V^i(\partial_i + \frac{x^i}{x^{n+1}}\partial_{n+1}), \sum_j^n V^j(\partial_j + \frac{x^j}{x^{n+1}}\partial_{n+1})) = \sum_i^n V^i V^j (\delta_{ij} - \frac{x^i x^j}{(x^{n+1})^2}) \tag{93}$$

$$= \sum_i^n (V^i)^2 - \frac{(xV)^2}{(x^{n+1})^2} \tag{94}$$

$$= |V|^2 - \frac{(x \cdot V)^2}{(x^{n+1})^2} \tag{95}$$

From Cauchy-Schwarz, we know that $(x \cdot V)^2 \leq |x|^2|V|^2 \leq (x^{n+1})^2|V|^2$. Thus, $\frac{(x\cdot V)^2}{(x^{n+1})} \leq |V^2|$ with equality if and only if $V = 0$, so g is in fact Riemannian.

Now we calculate curvature. We use Theorem 2.5 in Do Carmo:

$$K(x, y) - \overline{K}(x, y) = \langle B(x, x), B(y, y)\rangle - |B(x, y)|^2 \tag{96}$$

for orthonormal x and y. Let $x = \partial_i$ and $y = \partial_j$.

For the metric $\overline{g}_{ij} = \delta_{ij}$, we see that, since the metric is constant valued, all Christoffel symbols associated with $\overline{g}$ are 0, so $\overline{K}(x, y) = 0$. By definition, $B(x, y) = \overline{\nabla}_{\overline{x}}\overline{y} - \nabla_x y$. Since we are subtracting the tangent vector $\nabla_x y$ from $\overline{\nabla}_{\overline{x}}\overline{y}$, we are left with the projection of $\overline{\nabla}_{\overline{x}}\overline{y}$ onto some normal vector N: $B(x, y) = \frac{\langle \overline{\nabla}_{\overline{x}}\overline{y}, N\rangle}{\langle N, N\rangle}N$. To find N, we demand $\overline{g}((x^1, ..., x^n, \frac{\sum_i^n x^i}{x^{n+1}}), N) = 0$, where the first argument is our tangent vector from above. If we plug in $N = (x^1, ..., x^n, x^{n+1})$, we get this inner product equal to $\sum_i^n (x^i)^2 - x^{n+1}\frac{\sum_i^n (x^i)^2}{x^{n+1}} = 0$. We also see that $\overline{g}(N, N) = \sum_i^n (x^i)^2 - (x^{n+1})^2 = -1$. Now we have

$$K(x,y) = \langle \frac{\langle \overline{\nabla}_{\overline{x}}\overline{x}, N\rangle}{|N|^2}N, \frac{\langle \overline{\nabla}_{\overline{y}}\overline{y}, N\rangle}{|N|^2}N\rangle - |\frac{\langle \overline{\nabla}_{\overline{x}}\overline{y}, N\rangle}{\langle N, N\rangle}N|^2 \tag{97}$$

$$= \langle \frac{\langle \overline{x}, \overline{\nabla}_{\overline{x}}N\rangle}{|N|^2}N, \frac{\langle \overline{y}, \overline{\nabla}_{\overline{y}}N\rangle}{|N|^2}N\rangle - |\frac{\langle \overline{y}, \overline{\nabla}_{\overline{x}}N\rangle}{\langle N, N\rangle}N|^2 \tag{98}$$

$$\tag{99}$$

Due to compatibility of the metric. Also, we notice that $\overline{\nabla}_{\overline{x}}N = \overline{x}^i \frac{\partial N^j}{\partial \overline{x}^i}\partial_i = \overline{x}^i\partial_i = \overline{x}$, so we now have

$$K(x,y) = \langle \frac{\langle \overline{x}, \overline{x}\rangle}{-1}N, \frac{\langle \overline{y}, \overline{y}\rangle}{-1}N\rangle - |\frac{\langle \overline{y}, \overline{x}\rangle}{-1}N|^2 \tag{100}$$

$$= \langle -N, -N\rangle - 0 \tag{101}$$

$$= (-1)^2\langle N, N\rangle \tag{102}$$

$$= -1 \tag{103}$$

because $\overline{x}, \overline{y}$ are orthonormal. □