

Enumerative Geometry and String Theory

Exercises

Alec Lau

These are all the problems in the book of the title, written by Sheldon Katz.
Errata are my own unless otherwise noted.

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1 Warming up to Enumerative Geometry

Question 1.1. Use limits to verify the assertion that a root of a quadratic equation goes off to infinity as its leading coefficient a_0 approaches 0 while a_1 remains finite. Verify that two solutions go off to infinity as a_1 and a_0 both approach 0 (assuming a_2 stays nonzero).

Proof. Recall the quadratic formula giving roots

$$x = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_0}$$

As we take $a_0 \rightarrow 0$, we assume the discriminant $a_1^2 - 4a_0a_2$ is always positive, as a_0 eventually will become less than $\frac{a_1^2}{4a_2}$. If the discriminant is negative, then a_0a_2 , which must be positive, and will eventually decrease below $\frac{a_1^2}{4}$, making the discriminant positive, and thus making its square roots defined in $\mathbb{R}$. Thus we have two roots of our quadratic equation as $a_0 \rightarrow 0$.

The distance between the two roots is $\frac{2\sqrt{a_1^2 - 4a_0a_2}}{2|a_0|}$. The limit as $a_0 \rightarrow 0$ in the numerator is $2|a_1^2|$, which we've assumed is finite. Thus the limit of the distance between the roots is

$$\lim_{a_0 \rightarrow 0} \frac{1}{|a_0|} \sqrt{a_1^2 - 4a_0a_2} = \left(\lim_{a_0 \rightarrow 0} \frac{1}{|a_0|} \right) \left(\lim_{a_0 \rightarrow 0} \sqrt{a_1^2 - 4a_0a_2} \right) = \infty$$

The distance between the roots therefore goes off to infinity, so at least one root must go off to infinity.

When $(a_0, a_1) \rightarrow (0, 0)$, we first assume a_0 is sufficiently small. Then our roots can be approximated using a Taylor expansion:

$$\frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_0} = \frac{-a_1 \pm a_1(1 - \frac{4a_0a_2}{2a_1^2}) + \mathcal{O}(a_0^2)}{2a_0} = \frac{-a_1}{2a_0} \pm \left(\frac{a_1}{2a_0} - \frac{a_2}{a_1} \right) + \mathcal{O}(a_0^2)$$

Thus our roots become $-\frac{a_2}{a_1} + \mathcal{O}(a_0^2)$, $-\frac{a_1}{a_0} + \frac{a_2}{a_1} + \mathcal{O}(a_0^2)$. The limits as $a_1 \rightarrow 0$ of these two roots thus become $-\infty$ and ∞ , respectively. The limits as $a_0 \rightarrow 0$ of these two roots become $\frac{a_2}{a_1}$ and $-\infty$, which, then taking the limit for the first root as $a_1 \rightarrow 0$ puts the first limit as $-\infty$ and the second as ∞ . $\square$

Question 1.2. *Prove that any two equivalence classes $[a]$ and $[b]$, for a, b in a set S , are either equal or disjoint and that S is the union of all the equivalence classes.*

Proof. Suppose that we have two equivalence classes $[a]$ and $[b]$ in S that are not disjoint. Then there exists some element x in $[a]$ and $[b]$. Then, $x \sim a$ for all a in $[a]$ and $x \sim b$ for all b in $[b]$. By symmetry, $b \sim x, \forall b \in [b]$. Then $b \sim x \sim a, \forall a \in [a], b \sim [b]$. By transitivity, $b \sim a, \forall a \in [a], b \sim [b]$, and therefore $[a] = [b]$.

Suppose that S is not the union of all equivalence classes. Then there is an element $x \in S$ such that $x \in C, \forall C \subset S$. By reflexivity, $x \sim x$, so x is its own equivalence class, as $x \sim a \forall a \in [x]$. Thus all elements belong to an equivalence class within S . $\square$

Question 1.3. *Let V be a vector space over a field k and consider the relation on $V - \mathbf{0}$ defined by scalar multiplications: $v \sim v'$ if and only if $v = \lambda v'$ for some $\lambda \in k - \{0\}$. Prove that $\sim$ is an equivalence relation.*

Proof. We need to prove reflexivity, symmetry, and transitivity.

Reflexivity: We have $v \sim v$ because $v = 1v$, so $v \sim 1v = v$.

Symmetry: Suppose $v \sim v'$. Then $v = \lambda v'$ for $\lambda \in k - \{0\}$. Then $\frac{1}{\lambda}$ is well-defined, so $\frac{1}{\lambda}v = v'$, so $v' = \frac{1}{\lambda}v$. Since $\frac{1}{\lambda} \in k - \{0\}$, we therefore have $v' \sim v$.

Transitivity: If $v \sim v'$ and $v' \sim v''$, then we have $v = \lambda' v'$ and $v' = \lambda'' v''$ for $\lambda', \lambda'' \in k - \{0\}$. Then $v = \lambda' \lambda'' v''$, and since $\lambda' \lambda'' \in k - \{0\}$, $v \sim v''$. $\square$

Question 1.4. *Show that every nonzero homogeneous polynomial $F(x_0, x_1)$ of degree 2 has exactly two roots in $\mathbb{P}^1$ including multiplicity.*

Proof. Let $F(x_0, x_1) := a_0 x_0^2 + a_1 x_0 x_1 + a_2 x_1^2$.

In the case $a_2 = 0$, $F(x_0, x_1) = x_0(a_0 x_0 + a_1 x_1)$. In this case, the roots are $(0, 1)$ and $(a_1, -a_0)$.

Assume $a_2 \neq 0$. The dehomogenization of $F(x_0, x_1)$ gives $x_0^2 f(\frac{x_1}{x_0})$ for $f(x) := a_0 + a_1 x + a_2 x^2$. From the quadratic formula, we know therefore that the roots of

$F(x_0, x_1)$ are subject to

$$x_1 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_2}x_0$$

If the discriminant $\sqrt{a_1^2 - 4a_0a_2}$ is nonzero, then these are two distinct roots. If not, then we have $a_1^2 = 4a_0a_2$. In this case,

$$F(x_0, x_1) = (\sqrt{a_0}x_0 + \sqrt{a_2}x_1)^2$$

Thus we have one root with multiplicity 2 given by $(\sqrt{a_2}, -\sqrt{a_0})$. $\square$

Question 1.5. Let V be a vector space. Find a natural one-to-one correspondence between $\mathbb{P}^1(V)$ and the set of all 1-dimensional linear subspaces of V .

Proof. Label the ground field of V be K . Let V^1 be a 1-dimensional linear subspace of V , and $(v_0, v_1, \dots)$ be a basis vector of V^1 . Then $V^1 = \lambda(v_0, v_1, \dots), \forall \lambda \in K^*$. The correspondence between all V^1 and $\mathbb{P}(V)$ is then

$$\lambda(v_0, v_1, \dots) \mapsto \lambda(v_0, v_1, \dots) \in \mathbb{P}(V), \forall \lambda \in K^*$$

Note this is well-defined because all such basis vectors are identified in $\mathbb{P}(V)$.

Let $(v_0, v_1, \dots)$ be a point in $\mathbb{P}(V)$. Then let V^1 be the 1-dimensional subspace spanned by $(v_0, v_1, \dots)$, i.e. $V^1 = \lambda(v_0, v_1, \dots) \forall \lambda \in K^*$. If $(v'_0, v'_1, \dots)$ is another point in $\mathbb{P}(V)$ such that $(v_0, v_1, \dots) \neq (v'_0, v'_1, \dots)$, this means that there is no $\lambda' \in K^*$ such that $(v_0, v_1, \dots) = \lambda'(v'_0, v'_1, \dots)$. Therefore, $(v'_0, v'_1, \dots)$ is not in V^1 , so separate points in $\mathbb{P}(V)$ correspond to separate 1-dimensional subspaces of V . $\square$

Question 1.6. Show that $\mathbb{P}^n - U_0$ is in natural one-to-one correspondence with $\mathbb{P}^{n-1}$.

Proof. Recall that $U_0 := \{(x_0, \dots, x_n) \in \mathbb{P}^n | x_0 \neq 0\}$. Then $\mathbb{P}(V) - U_0 = \{(0, x_1, \dots, x_n) \in \mathbb{P}^n\}$. Consider the natural map

$$\mathbb{P} - U_0 \rightarrow \mathbb{P}^{n-1}$$

$$(0, x_1, \dots, x_n) \mapsto (x_1, \dots, x_n)$$

This is a bijection because an arbitrary point $(x_1, \dots, x_n) \in \mathbb{P}^{n-1}$ has preimage $(0, x_1, \dots, x_n) \in \mathbb{P}^n - U_0$, and if we have another point $(0, x'_1, \dots, x'_n) \in \mathbb{P}^n - U_0$ that maps to $(x_1, \dots, x_n)$, then $\lambda'(x_1, \dots, x_n) = (x'_1, \dots, x'_n)$ for some $\lambda' \in \mathbb{C}^*$. Then $(x'_1, \dots, x'_n)$ has preimage $(0, \lambda'x_1, \dots, \lambda'x_n)$ which is identified with $(0, x_1, \dots, x_n)$. $\square$

Question 1.7. Suppose that $F(x_0, x_1, \dots, x_n)$ is a nonzero homogeneous polynomial of degree d .

- a Prove that $F(\lambda x_0, \dots, \lambda x_n) = \lambda^d F(x_0, \dots, x_n)$ for any complex number λ .
- b Prove that the zero locus $Z(F) \subset \mathbb{P}^n$ is well defined; i.e., $F(\mathbf{x}) = 0$ if and only if $F(\lambda \mathbf{x}) = 0$ for any $\lambda \in \mathbb{C}^*$
- c Show that if F is a polynomial and $Z(F)$ is well defined in the sense of (b) above, then F must be homogeneous of some degree.

Proof. a F can be written as

$$\sum_{i_0, \dots, i_n}^{d, \dots, d} a_{i_0 \dots i_n} x_0^{i_0} \dots x_n^{i_n} \mid i_0 \geq 0, \dots, i_n \geq 0, i_0 + \dots + i_n = d$$

Then $F(\lambda x_0, \dots, \lambda x_n)$ is written as a sum of the form $a_{i_0 \dots i_n} \lambda^{i_0} x_0^{i_0} \dots \lambda^{i_n} x_n^{i_n}$. Since $i_0 + \dots + i_n = d$, this expression is equal to $a_{i_0 \dots i_n} \lambda^d x_0^{i_0} \dots x_n^{i_n}$. Since all such terms have λ^d , we can factor this from the sum, yielding $\lambda^d F(x_0, \dots, x_n)$.

- b If $F(\mathbf{x}) = 0$, then $F(\lambda \mathbf{x}) = \lambda^d F(\mathbf{x})$ as we proved above, and this is equal to $\lambda^d \cdot 0$. If $F(\lambda \mathbf{x}) = 0$ for any $\lambda \in \mathbb{C}^*$, then $0 = F(\lambda \mathbf{x}) = \lambda^d F(\mathbf{x})$ for any $\lambda \in \mathbb{C}^*$. This can only be equal to zero for any λ if $F(\mathbf{x}) = 0$.
- c If $Z(F) \subset \mathbb{P}^n$ is well-defined, then $F(\mathbf{x}) = 0$ if and only if $F(\lambda \mathbf{x}) = 0$ for any $\lambda \in \mathbb{C}^*$. We will express a general nonzero polynomial $F(\mathbf{x})$ as

$$\sum_{i_0, \dots, i_n}^{d_0, \dots, d_n} a_{i_0 \dots i_n} x_0^{i_0} \dots x_n^{i_n} \mid i_0 \geq 0, \dots, i_n \geq 0$$

Then

$$F(\lambda \mathbf{x}) = \lambda^m \sum_{i_0 \dots i_n} a_{i_0 \dots i_n} \lambda^{i_0 + \dots + i_n - m} x_0^{i_0} \dots x_n^{i_n}$$

where $m := \min(i_0 + \dots + i_n)$. If $\mathbf{x}$ is such that

$$F(\mathbf{x}) = \sum_{i_0, \dots, i_n} a_{i_0 \dots i_n} x_0^{i_0} \dots x_n^{i_n} = 0$$

the only way that $F(\lambda \mathbf{x}) = 0$ for any $\lambda \in \mathbb{C}^*$ is for $i_0 + \dots + i_n - m = 0$ for all terms $i_0, \dots, i_n$; therefore, $i_0 + \dots + i_n$ must be constant for all $i_0, \dots, i_n$ combinations, which means the powers of $x_0, \dots, x_n$ must all add to a constant number, i.e. F is homogeneous. □

Question 1.8. *Prove that any two distinct lines in $\mathbb{P}^2$ have exactly one point of intersection.*

Proof. We'll label two lines in $\mathbb{P}^2$ by the zero loci of $a_0x_0 + a_1x_1 + a_2x_2 = 0$ and $a'_0x'_0 + a'_1x'_1 + a'_2x'_2 = 0$. These intersect when there is a point (y_0, y_1, y_2) such that

$$\begin{aligned} a_0y_0 + a_1y_1 + a_2y_2 &= 0 \\ a'_0y_0 + a'_1y_1 + a'_2y_2 &= 0 \end{aligned}$$

or

$$\begin{pmatrix} a_0 & a_1 & a_2 \\ a'_0 & a'_1 & a'_2 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \mathbf{0}$$

Since the lines are assumed distinct, the matrix has rank 2. By the rank-nullity theorem, the null space, i.e. the (y_0, y_1, y_2) that satisfies the above equation, has dimension 1, i.e. this system's solution is the span of this point. In $\mathbb{P}^2$, this corresponds to a single point. □

2 Enumerative Geometry in the Projective Plane

Question 2.1. *Suppose that $f(x_1)$ is a polynomial and let $x_2 = f(x_1)$ be the equation defining its graph.*

- *Find the homogenization of this equation in $\mathbb{P}^2$. Let D be the corresponding (projective) plane curve.*
- *Find the equation for the tangent line to $y = f(x)$ at the point $(a, f(a))$; then find its homogenization to get a line $C \subset \mathbb{P}^2$.*
- *Use the working definition of the intersection multiplicity to show that $m_{(1,a,f(a))}(C, D) \geq 2$, where $(1, a, f(a))$ is the point of $\mathbb{P}^2$ corresponding to $(a, f(a)) \in \mathbb{C}^2$.*

- Show that if $(a, f(a))$ is a point of inflection, then

$$m_{(1,a,f(a))}(C, D) \geq 3.$$

Proof. • Let $F(x_0, x_1)$ be the homogenization of $f(x_1)$. With d the degree of $f(x)$, the homogenization of the equation $f(x_1) = x_2$ becomes

$$F(x_0, x_1) - x_2 x_0^{d-1} = 0$$

- The slope of the line tangent to $y = f(x)$ at $x = a$ is $\frac{df}{da}(a)$. Therefore the y-intercept is $f(a) - a \frac{df}{da}(a)$. Therefore the equation for the tangent line at $(a, f(a))$ is

$$\frac{df}{da}(a)(x_1 - a) + f(a) = x_2$$

Homogenizing this line yields

$$\frac{df}{da}(a)x_1 - x_2 + (f(a) - \frac{df}{da}(a)a)x_0 = 0$$

- Since $F(1, x_1) = f(x_1)$, the point $(1, a, f(a))$ is in D . Thus, when we factor D into a product of d lines, one of the products must be $(f(a)x_1 - ax_2)^k$ for some $k > 0$. It is straightforward to calculate that $(1, a, f(a))$ lies in C as well. This is the derivative of D at $(1, a, f(a))$. By the product rule, the derivative of D is a sum of terms containing the undifferentiated term $(f(a)x_1 - ax_2)^k$ and one term containing $k(f(a)x_1 - ax_2)^{k-1}$. The terms of the derivative with the undifferentiated $(f(a)x_1 - ax_2)^k$ become zero at $(a, f(a))$, so, in order for $(1, a, f(a))$ to indeed be a root of the tangent line, we must have $k - 1 > 0$ for this final term to vanish. Therefore the multiplicity of $(1, a, f(a))$ in D is ≥ 2 , i.e. $m_{(1,a,f(a))}(C, D) \geq 2$.
- If $(1, a, f(a))$ is an inflection point, then the second derivative of D must be 0 at $(1, a, f(a))$. Following the logic of the above, the second derivative is a sum of terms containing $(f(a)x_1 - ax_2)^k$ or $k(f(a)x_1 - ax_2)^{k-1}$, with one final term containing the factor $k(k-1)(f(a)x_1 - ax_2)^{k-2}$. Thus must vanish as well if $(1, a, f(a))$ is to be an inflection point, so we must have $k - 2 > 0$, i.e. $m_{(1,a,f(a))}(C, D) \geq 3$.

□

Question 2.2. Use linear algebra to show that the intersection of n general hyperplanes in $\mathbb{P}^n$ is one point.

Proof. We label the i^{th} hyperplane by $a_{i0}x_0 + \dots + a_{in}x_n = 0$. Then a point $(x_0, \dots, x_n)$ of their intersection must satisfy

$$\begin{pmatrix} a_{10} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n0} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{0}$$

Note that the coefficient matrix has n rows. In general, the n rows of this matrix are not linearly dependent, so the matrix, in general, has full rank. Therefore, by the rank-nullity theorem, the dimension of the null space is $n + 1 - n = 1$, i.e. the $(x_0, \dots, x_n) \in \mathbb{P}^n$ is 1-dimensional. This means that the solution is the span of a single variable $(x_0, \dots, x_n) \in \mathbb{P}^n$. The span of a single point is all identified to one point in $\mathbb{P}^n$, so the intersection is a single point in $\mathbb{P}^n$. $\square$

Question 2.3. Show that the set of plane curves of degree d is in one-to-one correspondence with $\mathbb{P}^{d(d+3)/2}$.

Proof. Degree d plane curves are given by the zero loci of polynomials that are sums of terms

$$a_{i_0 i_1 i_2} x_0^{i_0} x_1^{i_1} x_2^{i_2} \mid 0 \leq i_0, i_1, i_2 \leq d, i_0 + i_1 + i_2 = d$$

Thus the number of $a_{i_0 i_1 i_2}$ is given by the number of d -length terms containing x_0, x_1 , and x_2 (unordered, due to multiplication being abelian in $\mathbb{C}^n$). In other words, there are 3-choose- d -with-replacement such terms $a_{i_0 i_1 i_2}$ in any degree d plane curve.

$$\left(\binom{3}{d} \right) = \binom{3+d-1}{d} = \frac{(d+2)!}{d!2!} = \frac{(d+2)(d+1)}{2} = \frac{d(d+3)}{2} + 1$$

Thus there are $\frac{d(d+3)}{2} + 1$ terms $a_{i_0 i_1 i_2}$, parameterizing points in $(\mathbb{P}^{d(d+3)/2+1-1})^*$. $\square$

Question 2.4. Show that the set of plane curves of degree d passing through a fixed point p is a hyperplane in $\mathbb{P}^{d(d+3)/2}$ under the correspondence of the previous exercise.

Proof. The degree d plane curves that pass through a point $p := (y_0, y_1, y_2)$ are described by the set of $a_{i_0 i_1 i_2}$ such that

$$\sum_{i_0, i_1, i_2} a_{i_0 i_1 i_2} y_0^{i_0} y_1^{i_1} y_2^{i_2} = 0 \mid 0 \leq i_0, i_1, i_2 \leq d, i_0 + i_1 + i_2 = d$$

This is a sum of $d(d+3)/2 + 1$ terms. Letting the $a_{i_0 i_1 i_2}$ vary within the corresponding points in $\mathbb{P}^{d(d+3)/2}$, they become their own variables. Since each of these variables are of degree 1, this describes a hyperplane in $\mathbb{P}^{d(d+3)/2}$. $\square$

Question 2.5. *Show that there are infinitely many conics containing five points p_i if any four of the p_i lie on the same line.*

Proof. Suppose that four of our five p_i points lie in a line. We will label the line

$$l_0 x_0 + l_1 x_1 + l_2 x_2 = 0$$

To include the final out-of-plane point $p_j := (p_{j0}, p_{j1}, p_{j2})$ in a conic C , we select (l'_0, l'_1, l'_2) such that

$$l'_0 p_{j0} + l'_1 p_{j1} + l'_2 p_{j2} = 0$$

So a conic that contains all five p_i can be written as

$$C := (l_0 x_0 + l_1 x_1 + l_2 x_2)(l'_0 x_0 + l'_1 x_1 + l'_2 x_2) = 0$$

We have one free variable in (l'_0, l'_1, l'_2) , as the only requirement is

$$\frac{l'_0 p_{j0} + l'_1 p_{j1}}{-p_{j2}} = l'_2$$

i.e. for any choice of l'_0 , we can choose any value of l'_1 without loss of generality. Thus there are an infinite number of conics containing our five points. $\square$

Question 2.6. *Argue as for the case $d = 2$ that given $d(d+3)/2$ points of $\mathbb{P}^2$ in general position, there is a unique plane curve of degree d containing all these points.*

Proof. Suppose we have $d(d+3)/2$ points in $\mathbb{P}^2$ in general position. We will label the i^{th} point by $(p_{i,0}, p_{i,1}, p_{i,2})$. As we saw in the third problem in this chapter, a degree d plane curve is specified by a unique point $(a_0, \dots, a_{d(d+3)/2}) \in \mathbb{P}^{d(d+3)/2}$. To clean up notation, we will denote $d(d+3)/2$ by $\mathfrak{d}$. A plane curve that contains all $\mathfrak{d}$ points would satisfy

$$\begin{pmatrix} p_{1,0}^d & \cdots & p_{1,0}^{i_0} p_{1,1}^{i_1} p_{1,2}^{i_2} |_{i_0+i_1+i_2=d} & \cdots & p_{1,2}^d \\ \vdots & & \ddots & & \vdots \\ p_{\mathfrak{d},0}^d & \cdots & p_{\mathfrak{d},0}^{i_0} p_{\mathfrak{d},1}^{i_1} p_{\mathfrak{d},2}^{i_2} |_{i_0+i_1+i_2=d} & \cdots & p_{\mathfrak{d},2}^d \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_{\mathfrak{d}} \end{pmatrix} = \mathbf{0}$$

where each row of the matrix on the left is represented by all $\mathfrak{d}$ words made by d -length words of p_0, p_1 , and p_2 . These represent $\mathfrak{d}$ hyperplanes in $\mathbb{P}^{\mathfrak{d}}$, for which $(a_0, \dots, a_{\mathfrak{d}})$ represent their intersection.

As we saw four problems ago, in general position, the rows are linearly independent, so by the Rank-Nullity Theorem the $(a_0, \dots, a_{\mathfrak{d}})$ that satisfies this system is a single point of intersection of the $\mathfrak{d}$ hyperplanes. Therefore there is a single plane curve of degree d containing all p_i . $\square$

Question 2.7. *Show that any rational curve of degree 2 in $\mathbb{P}^2$ is a plane curve of degree 2. Thus the terminology of the degree of a rational plane curve is consistent with the terminology for the degree of a hypersurface.*

Proof. We will denote an arbitrary degree 2 rational curve in $\mathbb{P}^2$ by (f_0, f_1, f_2) , where f_0, f_1 , and f_2 are images of degree 2 homogeneous polynomials of (x_0, x_1) with no common factors. Then degree 2 polynomials of (f_0, f_1, f_2) can be written as

$$\lambda_{00}f_0^2 + \lambda_{01}f_0f_1 + \lambda_{02}f_0f_2 + \lambda_{11}f_1^2 + \lambda_{12}f_1f_2 + \lambda_{22}f_2^2$$

Putting this in terms of our original variables, this becomes

$$x_0^4g_{40} + x_0^3x_1g_{31} + x_0^2x_1^2g_{22} + x_0x_1^3g_{13} + x_1^4g_{04}$$

where $g_{..}$ are each linear functions of $\lambda_{00}, \lambda_{01}, \lambda_{02}, \lambda_{11}, \lambda_{12}$, and λ_{22} . The question is now whether there exist $\lambda_{00}, \lambda_{01}, \lambda_{02}, \lambda_{11}, \lambda_{12}$, and λ_{22} such that all of the $g_{..}$ become equal to 0. In other words, we'd like to find the intersection of the zero loci of five hyperplanes $g_{..}$ in $\mathbb{P}^5$. By Bezout's Theorem, there is at least one point of intersection between these hyperplanes. This point is a value of $(\lambda_{00}, \lambda_{01}, \lambda_{02}, \lambda_{11}, \lambda_{12}, \lambda_{22})$ where the degree 2 polynomial of (f_0, f_1, f_2) is equal to zero. Defining

$$(y_0, y_1, y_2) := \text{Im}(f_0(x_0, x_2), f_1(x_0, x_1), f_2(x_0, x_1)) \in \mathbb{P}^5$$

we have the existence of at least one degree 2 homogeneous polynomial

$$P_{\lambda_{00}\lambda_{01}\lambda_{02}\lambda_{11}\lambda_{12}\lambda_{22}}(y_0, y_1, y_2)$$

equal to 0, defining a plane curve. $\square$

Question 2.8. *Let x denote the row vector (x_0, x_1, x_2) .*

(a) Show that any plane curve C of degree 2 can be written as

$$\mathbf{x}A\mathbf{x}^T = 0$$

for some nonzero 3×3 symmetric matrix A , which is unique up to a scalar multiple.

(b) Show that C is an irreducible rational curve if and only if A is invertible

(c) Show that C is a union of two distinct lines if and only if A has rank 2.

(d) Show that C is a double line if and only if A has rank 1.

Proof. (a) Denote an arbitrary degree 2 plane curve C by

$$a_{00}x_0^2 + 2a_{01}x_0x_1 + a_{11}x_1^2 + 2a_{02}x_0x_2 + 2a_{12}x_1x_2 + a_{22}x_2^2 = 0$$

If we let

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{pmatrix}$$

This matrix is symmetric and is unique up to a scalar multiple, and we get C by setting $\mathbf{x}A\mathbf{x}^T = 0$ by inspection.

(b) For now I don't know how to prove this other than proving c) and d).

(c) If A has rank 2, then by the Rank-Nullity Theorem, the null space of A is one-dimensional. Therefore, there is a nonzero $\mathbf{y}$ such that $A\mathbf{y} = 0$. This means that there is a nonzero $\mathbf{y}$ such that

$$a_{00}y_0 + a_{01}y_1 + a_{02}y_2 = 0$$

$$a_{01}y_0 + a_{11}y_1 + a_{12}y_2 = 0$$

$$a_{02}y_0 + a_{12}y_1 + a_{22}y_2 = 0$$

which define lines. Since A has rank 2, we can write exactly one line as a sum of the other two lines. Note that if these other two lines are not distinct, then A 's rank would be 1. C is thus a union of the two remaining distinct lines.

Conversely, if C is a union of two distinct lines, then we can write C as

$$(a_0x_0 + a_1x_1 + a_2x_2)(b_0x_0 + b_1x_1 + b_2x_2) = 0$$

We then write A (in a way to preserve symmetry) as

$$\begin{pmatrix} a_0b_0 & \frac{1}{2}(a_0b_1 + a_1b_0) & \frac{1}{2}(a_0b_2 + a_2b_0) \\ \frac{1}{2}(a_0b_1 + a_1b_0) & a_1b_1 & \frac{1}{2}(a_1b_2 + a_2b_1) \\ \frac{1}{2}(a_0b_2 + a_2b_0) & \frac{1}{2}(a_1b_2 + a_2b_1) & a_2b_2 \end{pmatrix}$$

Notice that this is equal to

$$\frac{1}{2} \left[\begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} (a_0 \ a_1 \ a_2) + \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} (b_0 \ b_1 \ b_2) \right]$$

This is the sum of two matrices with rank one, as each matrix's columns are multiples of one another. Thus we have $\text{Rank}(A) \leq 2$. A cannot have rank 1, since we would need two scalars λ_0, λ_1 such that

$$\lambda_0 A_{00} + \lambda_1 A_{01} = A_{02}$$

$$\lambda_0 A_{10} + \lambda_1 A_{11} = A_{12}$$

$$\lambda_0 A_{20} + \lambda_1 A_{21} = A_{22}$$

which is an overdetermined system for which, if $(a_0, a_1, a_2) \neq (b_0, b_1, b_2)$, there is no solution. Thus A has rank 2.

- (d) If A has rank 1, then the columns are all scalar multiples of one another. If that is the case, in projective space, the columns are all equal. Due to the repeated entries in A , this means all the matrix entries are the same:

$$\begin{array}{ccccc} a_{00} & \Longrightarrow & a_{01} & \Longrightarrow & a_{02} \\ & \swarrow & & & \\ a_{01} & \Longrightarrow & a_{11} & \Longrightarrow & a_{12} \\ & & & \swarrow & \\ a_{02} & & a_{12} & \Longrightarrow & a_{22} \end{array}$$

Thus, C is expressed as

$$\begin{aligned} a_{00}(x_0^2 + 2x_0x_1 + x_1^2 + 2x_0x_2 + x_2^2 + 2x_1x_2) &= 0 \\ \Rightarrow a_{00}(x_0 + x_1 + x_2)^2 &= 0 \end{aligned}$$

which is a double line.

Conversely, if C is a double line, then we have

$$C = (a_0x_0 + a_1x_1 + a_2x_2)^2 = 0$$

$$\Rightarrow A = \begin{pmatrix} a_0^2 & a_0a_1 & a_0a_2 \\ a_0a_1 & a_1^2 & a_1a_2 \\ a_0a_2 & a_1a_2 & a_2^2 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \begin{pmatrix} a_0 & a_1 & a_2 \end{pmatrix}$$

Thus the columns of A are scalar multiples of one another, and therefore A has rank 1. □

3 Stable Maps and Enumerative Geometry

Question 3.1. *Show that if $f = (f_0, \dots, f_4)$ is a parametrization of a rational curve of degree d on X , then for any homogeneous polynomial $h(x_0, x_1)$ of degree h , the replacement of each of the f_i by hf_i gives a solution to*

$$h_k(\{a_{ij}\}) = 0$$

with parametrizing polynomials of degree $d + h$.

Proof. Recall from the text that X is defined as the zero locus of a homogeneous degree five polynomial $F(x_0, \dots, x_4)$. Let

$$f_j(x_0, x_1) := \sum_{i=0}^d a_{ij} x_0^i x_1^{d-i}, \quad h(x_0, x_1) := \sum_{m=0}^h c_m x_0^m x_1^{h-m}$$

Since f is on X , we have

$$F(f_0, \dots, f_4) = 0,$$

$$\Rightarrow \sum_{k=0}^{5d} h_k(\{a_{ij}\}) x_0^k x_1^{5d-k} = 0$$

We then have

$$hf_j = \sum_{i=0}^d \sum_{m=0}^h a_{ij} x_0^{i+m} x_1^{d+h-m-i}$$

Since F is homogeneous, we have

$$F(hf_0, \dots, hf_4) = h^5 F(f_0, \dots, f_4) = 0,$$

$$h^5 \sum_{k=0}^{5d} h_k(\{a_{ij}\}) x_0^k x_1^{5d-k} = 0$$

We will express $h(x_0, x_1)^5$ as

$$h^5 = \sum_{n=0}^{5h} g_n(\{c_m\}) x_0^n x_1^{5h-n}$$

where g_n are degree h polynomials in the c_m coefficients. Therefore $F(hf_0, \dots, hf_4) = 0$ given by

$$\sum_{p=0}^{5(h+d)} \left[\sum_{n+k=p} g_n(\{c_m\}) h_k(\{a_{ij}\}) \right]_{n \leq 5h, k \leq 5d} x_0^p x_1^{5(d+h)-p} = 0$$

The $[\sum_{n+k=p} g_n(\{c_m\}) h_k(\{a_{ij}\})]_{n \leq 5h, k \leq 5d}$ are degree $h + d$ polynomials in c_m and a_{ij} , and each j value gives a parametrizing degree $5(d + h)$ polynomial. $\square$

Question 3.2. Show that the morphism $f : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ defined by $f(x_0, x_1) = (x_0^2, x_1^2, 0)$ is two-to-one onto the line $x_2 = 0$, except that it is one-to-one over the branch points $(1, 0, 0), (0, 1, 0)$.

Proof. Notice that (x_0, x_1) and $(x_0, -x_1)$ will be distinct points in $\mathbb{P}^1$ except when $x_0 = 0$ or $x_1 = 0$, yet these both map to $(x_0^2, x_1^2, 0)$. Thus f is two-to-one on $\mathbb{P}^1 \setminus (x_0, 0) \cup (0, x_1)$. At the point $(1, 0)$, these points are identified, and so $f(1, 0) = (1, 0, 0)$ is one-to-one. At the point $(0, 1)$, these points are also identified, so $f(0, 1) = (0, 1, 0)$ is one-to-one as well. $\square$

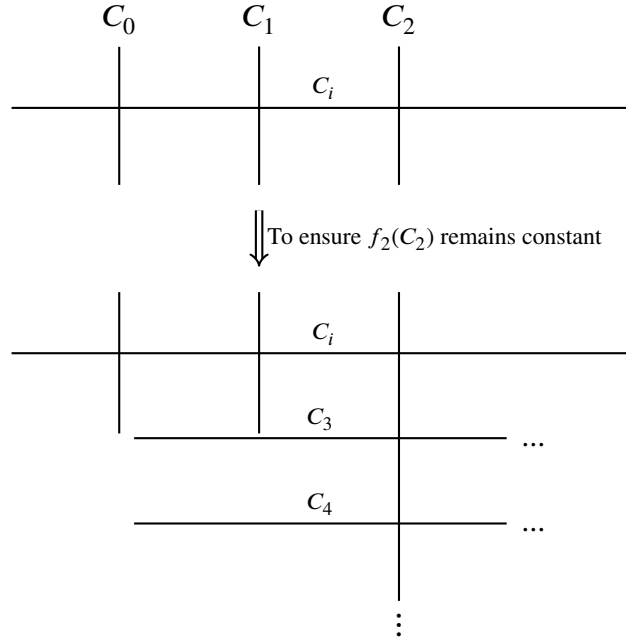
Question 3.3. Show that if $f \in \overline{M}(\mathbb{P}^2, 2)$, then the restriction of f to each component is nonconstant.

Proof. This is the moduli space of degree 2 stable maps into $\mathbb{P}^2$. Degree 2 stable maps are defined by a map $f : C \rightarrow \mathbb{P}^2$ such that each

$$f_i := f \circ \phi_i : \mathbb{P}^1 \rightarrow \mathbb{P}^2$$

is a parametrized rational curve or a constant map, with $\sum_i \deg(f_i) = 2$, and, if $f|_{C_i}$ is constant, then C_i is required to have at least 3 nodes of C .

Suppose that f_i is constant over C_i for some i . Then C_i is required to contain at least three nodes, i.e. it must intersect with at least three components, which we'll label C_0, C_1 , and C_2 . In order to keep $\deg(f) = 2$, we must have at least one of f_0, f_1, f_2 be degree 0, or constant, as well. Therefore, the corresponding component must intersect with at least two more components, C_3 and C_4 . Note that these components must be distinct from C_i, C_0, C_1 , and C_2 , since, if not, C would contain a cycle. However, f_3 and f_4 must also be constant to keep $\deg(f) = 2$, implying the existence of more distinct components. See the below diagram.



This recurses infinitely. This is a contradiction, however, since C must have finitely many components by definition. Thus there cannot be any such f_i . $\square$

Question 3.4. Show that if $f : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ has degree 2, then either f is a one-to-one correspondence onto a smooth conic or is two-to-one with two branch points onto a line.

Proof. By definition of morphisms, f must be a parametrized rational curve of degree 2. We showed in Exercise 7 of Chapter 2 that any rational curve of degree 2 in $\mathbb{P}^2$ is a plane curve of degree 2, i.e. a conic. Let

$$\alpha_i(x_0, x_1) := a_{00}^i x_0^2 + a_{01}^i x_0 x_1 + a_{11}^i x_1^2$$

Suppose we have two points $(x_0, x_1), (x'_0, x'_1)$ such that $f(x_0, x_1) = f(x'_0, x'_1)$. Then we have

$$\begin{pmatrix} a_{00}^0 & a_{01}^0 & a_{11}^0 \\ a_{00}^1 & a_{01}^1 & a_{11}^1 \\ a_{00}^2 & a_{01}^2 & a_{11}^2 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \mathbf{0} \quad (1)$$

where $(y_0, y_1, y_2) := (x_0^2 - x_0'^2, x_0x_1 - x_0'x_1', x_1^2 - x_1'^2)$. Suppose we have a smooth conic. Then all $\alpha_i \neq 0$. Since this is a parametrized rational curve, this matrix has full rank. Thus the null space is zero dimensional, so (y_0, y_1, y_2) must be the zero vector, i.e. $(x_0, x_1) = \pm(x'_0, x'_1)$. Therefore, if f is a smooth conic, then f is one-to-one.

Conics can either be smooth conics, line pairs, or double lines. A line pair has two components as shown in Example 3.3, which goes against our assumption, so instead suppose that our conic is a double line. Note that the image of f must therefore contain a zero, as the zero polynomial $\alpha_*(x_0, x_1) = 0$ is the only homogeneous polynomial that is constant. Without loss of generality, let our line in $\mathbb{P}^2$ be

$$(\alpha_0(x_0, x_1), \alpha_1(x_0, x_1), 0)$$

We can assume also that at least one of $\alpha_i \neq 0$, so without loss of generality assume $\alpha_1 \neq 0$. Subtracting $\frac{\alpha_0}{\alpha_1}$ times the second row from the first row, we get the equation

$$(a_{00}^0 - \frac{\alpha_0}{\alpha_1} a_{00}^1)(x_0 - x'_0)(x_0 + x'_0) = (a_{00}^0 - \frac{\alpha_0}{\alpha_1} a_{11}^1)(x_1 - x'_1)(x_1 + x'_1)$$

Since we assume general polynomials, we must have $x_0 = \pm x'_0$ and $x_1 = \pm x'_1$. Thus we have (x_0, x_1) and $\pm(x_0, -x_1)$ mapping to the same point on our double line. In the case where $x_0 \neq 0$ or $x_1 \neq 0$, these are distinct points, so the mapping is two-to-one. These become the same point at both $(1, 0)$ and $(0, 1)$, so the mapping then becomes one-to-one. Thus without loss of generality we have two branch points on this line: $(\alpha_0(1, 0), \alpha_1(1, 0), 0)$ and $(\alpha_0(0, 1), \alpha_1(0, 1), 0)$. $\square$

Question 3.5. Generalizing Exercise 3 from Chapter 2, prove that the set of hyper-surfaces of degree d in $\mathbb{P}^n$ is in one-to-one correspondence with $\mathbb{P}^{\binom{d+n}{d}-1}$. Note: the question in the book states $\mathbb{P}^{\binom{d+n}{n}-1}$, but I think this is a typo.

Proof. In $\mathbb{P}^n$, we have $n + 1$ variables in our polynomials. For degree d polynomials, we have $\binom{n+1}{d}$ possible combinations of d variables with replacement.

$$\left(\binom{n+1}{d} \right) = \binom{n+1+d-1}{d}$$

Thus each degree d hypersurface in $\mathbb{P}^n$ is uniquely described by $\binom{n+d}{d}$ coefficients for our $\binom{n+d}{d}$ terms modulo scalar multiplication for every term. $\binom{n+d}{d}$ -tuples modulo scalar multiplication are in one-to-one correspondence with $\binom{n+d}{d}$ coordinates of $\mathbb{P}^{\binom{n+d}{d}-1}$. $\square$

Question 3.6. Show that if $C \subset \mathbb{P}^2$ is a smooth conic, then $C^* \subset (\mathbb{P}^2)^*$ is a smooth conic. Show that $(C^*)^* = C$ under the natural identification $((\mathbb{P}^2)^*)^* = \mathbb{P}^2$.

Proof. $C^* \subset (\mathbb{P}^2)^*$ is the set of lines in $\mathbb{P}^2$ (set of elements in $(\mathbb{P}^2)^*$) such that $C \cap L$ is a single point with $m_p(C \cdot L) = 2$ for $L \in (\mathbb{P}^2)^*$. Choose an arbitrary point $p := (p_0, p_1, p_2) \in C$. The tangent line to C at p is

$$\frac{\partial C(p)}{\partial x_0} x_0 + \frac{\partial C(p)}{\partial x_1} x_1 + \frac{\partial C(p)}{\partial x_2} x_2 = 0$$

Define our smooth conic C by

$$C := a_{00}x_0^2 + a_{01}x_0x_1 + a_{11}x_1^2 + a_{02}x_0x_2 + a_{12}x_1x_2 + a_{22}x_2^2 = 0$$

The tangent line at p satisfies

$$\begin{aligned} & (2a_{00}p_0 + a_{01}p_1 + a_{02}p_2)x_0 + (2a_{11}p_1 + a_{01}p_0 + a_{12}p_2)x_1 + \\ & \quad (2a_{22}p_2 + a_{02}p_0 + a_{12}p_1)x_2 = 0 \\ & := a'_0(p_0, p_1, p_2)x_0 + a'_1(p_0, p_1, p_2)x_1 + a'_2(p_0, p_1, p_2)x_2 = 0 \end{aligned}$$

Reframing this, we have

$$\begin{pmatrix} 2a_{00} & a_{01} & a_{02} \\ a_{01} & 2a_{11} & a_{12} \\ a_{02} & a_{12} & 2a_{22} \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} = \mathbf{0}$$

Since C is a smooth conic, this matrix has full rank as shown in Chapter 2. Thus we can isolate each p_i as a linear function of a'_0, a'_1 , and a'_2 :

$$p_i := \lambda_{i0}a'_0 + \lambda_{i1}a'_1 + \lambda_{i2}a'_2$$

Since we must also have

$$a'_0 p_0 + a'_1 p_1 + a'_2 p_2 = 0$$

for our tangent line to contain p . Therefore $(a'_0, a'_1, a'_2) \in (\mathbb{P}^2)^*$, and we can also write:

$$\begin{aligned} & a'_0(\lambda_{00}a'_0 + \lambda_{01}a'_1 + \lambda_{02}a'_2) + a'_1(\lambda_{10}a'_0 + \lambda_{11}a'_1 + \lambda_{12}a'_2) \\ & \quad + a'_2(\lambda_{20}a'_0 + \lambda_{21}a'_1 + \lambda_{22}a'_2) = 0 \\ \Rightarrow & \lambda_{00}a'^2_0 + (\lambda_{01} + \lambda_{10})a'_0a'_1 + \lambda_{11}a'^2_1 + (\lambda_{02} + \lambda_{20})a'_0a'_2 \\ & \quad + (\lambda_{12} + \lambda_{21})a'_1a'_2 + \lambda_{22}a'^2_2 = 0 \end{aligned}$$

This is our (general) dual conic C^* , which is an expression of a homogeneous degree 2 polynomial of lines $(a'_0, a'_1, a'_2) \in (\mathbb{P}^2)^*$, i.e. a homogeneous degree 2 polynomial in $(\mathbb{P}^2)^*$.

Assume this is a line pair. Without loss of generality, assume then that $\lambda_{02} = \lambda_{20} = \lambda_{12} = \lambda_{21} = \lambda_{22} = 0$. Then p_0 is always 0, so C is a line pair, which contradicts our initial assumption. A similar contradiction arises if we assume C^* is a double line. Thus C^* is a smooth conic in $(\mathbb{P}^2)^*$.

To find $(C^*)^*$, we do the same process, resulting in a smooth conic of variables b'_0, b'_1 , and b'_2 such that

$$b'_0 a'_0 + b'_1 a'_1 + b'_2 a'_2 = 0$$

Since, for a given $(a'_0, a'_1, a'_2) \in (\mathbb{P}^2)^*$, we already have a tuple such that the above expression is 0, which is (p_0, p_1, p_2) , because that was how (a_0, a_1, a_2) was defined. Thus we have a natural identification $(b_0, b_1, b_2) \in ((\mathbb{P}^2)^*)^* \rightarrow (p_0, p_1, p_2) \in \mathbb{P}^2$. $\square$

Question 3.7. Suppose that C is a smooth conic. Prove that $p \in C$ if and only if L_p is tangent to C^* .

Proof. Recall $L_p \subset (\mathbb{P}^2)^*$ is the set of lines through p .

Suppose $p := (p_0, p_1, p_2) \in C$. We also have that $L_p = (p_0, p_1, p_2) \in ((\mathbb{P}^2)^*)^*$. Let our dual curve be

$$C^* = \lambda_{00}a'^2_0 + (\lambda_{01} + \lambda_{10})a'_0a'_1 + \lambda_{11}a'^2_1 + (\lambda_{02} + \lambda_{20})a'_0a'_2 + (\lambda_{12} + \lambda_{21})a'_1a'_2 + \lambda_{22}a'^2_2 = 0$$

where $\lambda_{..}$ are defined by $p_i := \lambda_{i0}a'_0 + \lambda_{i1}a'_1 + \lambda_{i2}a'_2$ as in the previous proof. Then the tangent line at (a'_0, a'_1, a'_2) is given by

$$\begin{aligned} & \begin{pmatrix} 2\lambda_{00}a'_0 + (\lambda_{01} + \lambda_{10})a'_1 + (\lambda_{02} + \lambda_{20})a'_2 \\ 2\lambda_{11}a'_1 + (\lambda_{01} + \lambda_{10})a'_0 + (\lambda_{12} + \lambda_{21})a'_2 \\ 2\lambda_{22}a'_2 + (\lambda_{02} + \lambda_{20})a'_0 + (\lambda_{12} + \lambda_{21})a'_1 \end{pmatrix} \cdot \begin{pmatrix} a'_0 \\ a'_1 \\ a'_2 \end{pmatrix} = 0 \\ &= \begin{pmatrix} p_0 + (\lambda_{00}a'_0 + \lambda_{10}a'_1 + \lambda_{20}a'_2) \\ p_1 + (\lambda_{01}a'_0 + \lambda_{11}a'_1 + \lambda_{21}a'_2) \\ p_2 + (\lambda_{02}a'_0 + \lambda_{12}a'_1 + \lambda_{22}a'_2) \end{pmatrix} \cdot \begin{pmatrix} a'_0 \\ a'_1 \\ a'_2 \end{pmatrix} \\ &= 2(p_0a'_0 + p_1a'_1 + p_2a'_2) = 0 \end{aligned}$$

This is the tangent line L_p .

Conversely, suppose that $L_p := (p_0, p_1, p_2) \in ((\mathbb{P}^2)^*)^*$ is a line tangent to C^* . Then $p_0a'_0 + p_1a'_1 + p_2a'_2 = 0$ for some $(a'_0, a'_1, a'_2) \in C^*$. By definition, (a'_0, a'_1, a'_2) is a line tangent to C , so we have

$$\begin{pmatrix} \frac{\partial C}{\partial p'_0}(p'_0) \\ \frac{\partial C}{\partial p'_1}(p'_1) \\ \frac{\partial C}{\partial p'_2}(p'_2) \end{pmatrix} \cdot \begin{pmatrix} p'_0 \\ p'_1 \\ p'_2 \end{pmatrix} = 0 = \begin{pmatrix} a'_0 \\ a'_1 \\ a'_2 \end{pmatrix} \cdot \begin{pmatrix} p'_0 \\ p'_1 \\ p'_2 \end{pmatrix}$$

for some $(p'_0, p'_1, p'_2) \in C$. Since $a'_0p'_0 + a'_1p'_1 + a'_2p'_2 = 0$ is a linear equation, its solution is uniquely determined up to a scalar factor, so $(p'_0, p'_1, p'_2) = (p_0, p_1, p_2) \in \mathbb{P}^2$, therefore p is in C . $\square$

4 Crash Course in Topology and Manifolds

Question 4.1. Let $a \in \mathbb{R}^n$ and $r > 0$. Let $B(a; r)$ denote the open ball in $\mathbb{R}^n$ of radius r centered at a :

$$B(a; r) = \{x \in \mathbb{R}^n \mid |x - a| < r\}.$$

Let $\mathcal{T}$ be the set of all arbitrary unions of balls $B(a; r)$. Show that $\mathcal{T}$ defined this way is a topology.

Proof. 1. Arbitrary unions of balls includes the empty set, so $\mathcal{T}$ contains the empty set.

2. Define $U := \bigcup_{i \in \mathbb{N}} B(0, i)$. U is an arbitrary union. Suppose let r_0 be an arbitrary element in $\mathbb{R}^n$. Let $n_{r_0} = |r_0|$. Since $\mathbb{N}$ goes on infinitely, there is always some $n \in \mathbb{N}$ such that $n > n_{r_0}$. Thus $r \in B(0, n)$, and therefore $r \in U$. Since this did not depend on the choice of r , we have that U covers all of $\mathbb{R}^n$, so $\mathbb{R}^n \in \mathcal{T}$.
3. If $\{U_\alpha\}$ is a set of elements of $\mathcal{T}$, they are a set of arbitrary unions of balls. Then their union $\bigcup_\alpha U_\alpha$ is also a union of balls, so $\bigcup_\alpha U_\alpha \in \mathcal{T}$.
4. Suppose we have two $U_1, U_2 \in \mathcal{T}$. If either of them is a union of balls given by an infinite diverging sequence of rs , then by a similar argument above, this element of the topology is $\mathbb{R}^n$, and therefore $\mathbb{R}^n \cap U = U$ for any element $U \in \mathcal{T}$, so $U_1 \cap U_2 \in \mathcal{T}$. If instead both are given by either an infinite converging sequences of rs or a finite sequence of rs , then let $r_{1_{\max}}$ be the maximum radius of the balls in U_1 (or the limit of the infinite converging sequence of rs if the limit is larger than all rs in U_1), and let $r_{2_{\max}}$ be defined the same way but for U_2 . Then $U_1 \cap U_2$ is $B(a; \min(r_{1_{\max}}, r_{2_{\max}}))$. The set $\{B(a; \min(r_{1_{\max}}, r_{2_{\max}}))\}$ is an arbitrary union of balls $B(a; r), r > 0$, and so is an element of $\mathcal{T}$.

□

Question 4.2. Let X be a topological space, and let $Y \subset X$ be a subset. Let

$$\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}_X\}$$

Prove that $\mathcal{T}_Y$ is a topology.

- Proof.*
1. Since $\mathcal{T}_X$ is a topology, $\emptyset \in \mathcal{T}_X$, and $\emptyset \cap Y = \emptyset$, so $\emptyset \in \mathcal{T}_Y$.
 2. Since $\mathcal{T}_X$ is a topology, $X \in \mathcal{T}_X$, and $X \cap Y = Y$ since $Y \subset X$, so $Y \in \mathcal{T}_Y$.
 3. Each set $U_\alpha \in \mathcal{T}_Y$ has a corresponding set $U'_\alpha \in \mathcal{T}_X$ such that $U'_\alpha \cap Y = U_\alpha$. If $\{U_\alpha\}$ is a set of elements of $\mathcal{T}_Y$, let $\{U'_\alpha\}$ be the corresponding set of elements of $\mathcal{T}_X$ such that $U'_\alpha \cap Y = U_\alpha$. $\bigcup_\alpha U_\alpha = \bigcup_\alpha (U'_\alpha \cap Y) = (\bigcup_\alpha U'_\alpha) \cap Y$. Since $\mathcal{T}_X$ is a topology, $\bigcup_\alpha U'_\alpha \in \mathcal{T}_X$, so $\bigcup_\alpha U_\alpha \in \mathcal{T}_Y$.
 4. If U_1, U_2 are elements of $\mathcal{T}_Y$, let U'_1, U'_2 be the elements in $\mathcal{T}_X$ such that $U'_1 \cap Y = U_1, U'_2 \cap Y = U_2$. Because $\mathcal{T}_X$ is a topology, $U'_1 \cap U'_2 \in \mathcal{T}_X$, so $(U'_1 \cap U'_2) \cap Y \in \mathcal{T}_Y$, so $U_1 \cap U_2 \in \mathcal{T}_Y$.

□

Question 4.3. *Prove that a closed subset of a compact topological space is compact in the subspace topology.*

Proof. Let $U' \subset X$ be an arbitrary closed subset, and define $U := X - U'$ be the complement of U' in X 's topology. Let $\{U'_\alpha\}_{\alpha \in I'}$ be an arbitrary open cover of U' . Then

$$\{U'_\alpha\}_{\alpha \in I'} \cup U = X$$

$\{U'_\alpha\}_{\alpha \in I'} \cup U$ is an open cover of X . Since X is compact, there must exist a finite subcover of this open cover, so there must exist a finite $J' \subset I'$ such that

$$\{U'_\alpha\}_{\alpha \in J'} \cup U = X$$

and therefore $\{U'_\alpha\}_{\alpha \in J'}$ is a finite subcover of $\{U'_\alpha\}_{\alpha \in I'}$ of U' . □

Question 4.4. (a) *Show that isomorphism in the category of sets is the usual notion of bijection.*

(b) *Show that isomorphism in the category of vector spaces is the usual notion of an invertible linear transformation.*

Proof. (a) Let $f : X \rightarrow Y$ be a bijection between sets X and Y . Because f is a bijection, it is onto, so every element $y \in Y$ has a preimage $f^{-1}(y) \in X$, and since f is one-to-one, the preimage is unique. Therefore, for every $x \in X$, $f^{-1}(f(x)) = x$. Similarly, for every $y \in Y$, $f(f^{-1}(y)) = y$.

(b) We will represent an invertible linear transformation between two vector spaces $F : V \rightarrow W$ by definition has an inverse linear transformation F^{-1} such that $F^{-1}F = I_V$, the identity transformation on V . Furthermore, since $F^{-1}FV = I_V V = V$, we have

$$\begin{aligned} F^{-1}FV &= I_V V \\ \Rightarrow FF^{-1}FV &= FI_V V \\ \Rightarrow FF^{-1}W &= FV \\ \Rightarrow FF^{-1}W &= W \end{aligned}$$

so $FF^{-1} = I_W$, the identity transformation on W . □

Question 4.5. If a function $f : \mathbb{R} \rightarrow \mathbb{R}$ has the property that, given any $a \in \mathbb{R}$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$, then f is continuous in the sense of topology; namely, $f^{-1}(V)$ is open for all open sets V .

Proof. Fix an $a \in \mathbb{R}$. Define our topology on our $\mathbb{R}$ codomain as the set of all arbitrary unions of open balls $B(f(a), r)$ for $r > 0$, and define the topology on our $\mathbb{R}$ domain as the set of all arbitrary unions of open balls $B(a, r)$ defined the same way. Then, on our assumption, we have $f^{-1}(B(f(a), \varepsilon)) = B(a, \delta)$, for each $B(f(a), \varepsilon) \subset \mathbb{R}$. We also have $f^{-1}(B(f(a), \varepsilon_1) \cup B(f(a), \varepsilon_2)) = B(a, \delta_1) \cup B(a, \delta_2)$, so this is closed under unions. Hence $f^{-1}(V)$ is open for all open sets V . $\square$

Question 4.6. For $i = 0, \dots, n$ define $U_i = \{(x_0, \dots, x_n) \in \mathbb{P}^n | x_i \neq 0\}$, and the maps

$$\phi_i : U_i \rightarrow \mathbb{C}^n, (x_0, \dots, x_n) \mapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i} \right) \rightarrow \mathbb{R}^{2n}$$

where the final identification is given by

$$(y_1, \dots, y_n) \mapsto (\operatorname{Re}(y_1), \operatorname{Im}(y_1), \dots, \operatorname{Re}(y_n), \operatorname{Im}(y_n))$$

A subset $U \subset \mathbb{P}^n$ is open if $\phi_i(U \cap U_i)$ is open in $\mathbb{C}^n$ for all i . Show that these sets define a topology and that the ϕ_i are homeomorphisms. Hence $\mathbb{P}^n$ is a manifold.

Proof. When U is the empty set, it is mapped to the empty set in $\mathbb{R}^{2n}$'s topology, which is an open set, so $\emptyset$ is included. When $U = \mathbb{P}^n$, we have $\phi_i(U \cap U_i) = \phi_i(U_i)$. Because division by a nonzero scalar is continuous, by the previous problem we have that $\phi_i(U_i)$ is open in $\mathbb{R}^{2n}$. Note that, for transition functions ψ_{i1} , we have

$$\begin{aligned} \psi_{i1} : \mathbb{C}^n &\rightarrow \mathbb{C}^n \\ (c_0, \dots, c_n) &\mapsto \left(\frac{1}{c_i}, \frac{c_0}{c_i}, \dots, \frac{c_{i-1}}{c_i}, \frac{c_{i+1}}{c_i}, \dots, \frac{c_n}{c_i} \right) \end{aligned}$$

which, due to multiplication and division by nonzero scalars, is a homeomorphism. The other transition functions ψ_{ij} can be shown to be homeomorphic in the same way.

If we have two open sets U_1, U_2 , we have $\phi_i((U_1 \cup U_2) \cap U_i) = \phi_i((U_1 \cap U_i) \cup (U_2 \cap U_i))$. This is the union of two open sets in $\mathbb{R}^{2n}$, which is itself another

open set of $\mathbb{R}^{2n}$. Thus $U_1 \cup U_2$ is an open set as well. We also have $\phi_i((U_1 \cap U_2) \cap U_i) = \phi_i(U_1 \cap U_i) \cap \phi_i(U_2 \cap U_i)$, which is an open set of $\mathbb{R}^{2n}$, so $U_1 \cap U_2$ is open as well. Hence this defines a topology.

We examine the following:

$$\begin{aligned}
& \phi_0^{-1}(\phi_0(a_0 + ib_0, \dots, a_n + ib_n)) \\
&= \phi_0^{-1}\left(\frac{a_1 + ib_1}{a_0 + ib_0}, \dots, \frac{a_n + ib_n}{a_0 + ib_0}\right) \\
&= \phi_0^{-1}\left(\frac{a_1 a_0 + b_1 b_0}{a_0^2 + b_0^2}, \frac{b_1 a_0 - a_1 b_0}{a_0^2 + b_0^2}, \dots, \frac{a_n a_0 + b_n b_0}{a_0^2 + b_0^2}, \frac{b_n a_0 - a_n b_0}{a_0^2 + b_0^2}\right) \\
&= \left(1, \frac{a_1 + ib_1}{a_0 + ib_0}, \dots, \frac{a_n + ib_n}{a_0 + ib_0}\right) \\
&= (a_0 + ib_0, a_1 + ib_1, \dots, a_n + ib_n) \\
& \phi_0(\phi_0^{-1}(a_0, b_0, \dots, a_n, b_n)) \\
&= \phi_0(1, a_0 + ib_0, \dots, a_n + ib_n) \\
&= (a_0, b_0, \dots, a_n, b_n)
\end{aligned}$$

Every operation involved here is either multiplication by a scalar, addition, or division by a nonzero scalar. These are all continuous in the sense of calculus, and therefore from the previous problem, continuous in topology. The other ϕ_i maps work in the same way. Thus $\phi_i^{-1}\phi_i = Id_{\mathbb{P}^n}$ and $\phi_i(\phi_i^{-1}) = Id_{\mathbb{R}^{2n}}$, and the ϕ_i are therefore homeomorphisms. $\square$

Question 4.7. Suppose that X is a set covered by a collection of subsets $\{U_\alpha\}$. Suppose we are given one-to-one correspondences $\phi_\alpha : U_\alpha \rightarrow V_\alpha$, such that the maps $\psi_{\alpha\beta}$ defined via

$$\begin{aligned}
\psi_{\alpha\beta} &: \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \phi_\alpha(U_\alpha \cap U_\beta), \\
\psi_{\alpha\beta} &= \phi_\alpha \circ (\phi_\beta|_{U_\alpha \cap U_\beta})^{-1}
\end{aligned}$$

are homeomorphisms. Show that there is a natural topology on X such that the $\{(U_\alpha, \phi_\alpha)\}$ form an atlas.

Proof. Note: in the text, V_α are open sets in $\mathbb{R}^n$. Define $\mathcal{T}_X$ by the collection of subsets $U \subseteq X$ such that $\phi_\alpha(U \cap U_\alpha)$ is open in $\mathbb{R}^n$ for all α . We have $\emptyset \in \mathcal{T}$, as $\phi_\alpha(\emptyset \cap U_\alpha) = \phi_\alpha(\emptyset) = \emptyset$ for all α , which is open in $\mathbb{R}^n$'s topology. We also have $X \in \mathcal{T}$, since $\phi_\alpha(X \cap U_\alpha) = \phi_\alpha(U_\alpha) = V_\alpha$ for all α , which are open in

$\mathbb{R}^n$ by definition. If we have two $U_1, U_2 \subset X$ in this collection, this means that $\phi_\alpha(U_1 \cap U_\alpha)$ and $\phi_\alpha(U_2 \cap U_\alpha)$ are open sets for all α . Therefore $\phi_\alpha((U_1 \cup U_2) \cap U_\alpha) = \phi_\alpha(U_1 \cap U_\alpha) \cup \phi_\alpha(U_2 \cap U_\alpha)$ for all α . Since $\mathbb{R}^n$'s topology is closed under unions, this is also an open set. Similar logic follows for intersections. Thus $\mathcal{T}_X$ defines a topology on X . Since the $\psi_{\alpha\beta}$ are homeomorphisms, our $\{\phi_\alpha\}$ define an atlas. $\square$

Question 4.8. *Prove that the n -sphere S^n is an n -dimensional manifold.*

Proof. S^n is defined as $\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \dots + x_n^2 = 1\}$, so it is a topological space with the subspace topology. Define

$$U_1 := S^n - (0, \dots, 0, 1)$$

$$U_2 := S^n - (0, \dots, 0, -1)$$

and let our charts be stereographic projections:

$$\begin{aligned} \phi_1 : U_1 &\rightarrow \mathbb{R}^n \\ (x_0, \dots, x_n) &\mapsto \left(\frac{x_0}{1-x_n}, \dots, \frac{x_{n-1}}{1-x_n} \right) \\ \phi_1^{-1}(x_0, \dots, x_{n-1}) &= \left(\frac{2x_0}{|x|^2 + 1}, \dots, \frac{2x_{n-1}}{|x|^2 + 1}, \frac{-2}{|x|^2 + 1} + 1 \right) \\ \phi_2 : U_2 &\rightarrow \mathbb{R}^n \\ (x_0, \dots, x_n) &\mapsto \left(\frac{x_0}{1+x_n}, \dots, \frac{x_{n-1}}{1+x_n} \right) \\ \phi_2^{-1}(x_0, \dots, x_{n-1}) &= \left(\frac{2x_0}{|x|^2 + 1}, \dots, \frac{2x_{n-1}}{|x|^2 + 1}, \frac{2}{|x|^2 + 1} - 1 \right) \end{aligned}$$

where we found these formulas by using point-slope form and intersecting the resulting line with the sphere. Both of these charts are only multiplying/dividing by nonzero scalars, and so are both continuous, and therefore their transition maps are homeomorphic. Thus S^n is an n -manifold given this atlas. $\square$

Question 4.9. *Convince yourself that a surface of genus g is a manifold of dimension 2.*

Proof. Since no proof is required, this'll be very ad hoc. A genus g surface is $g+1$ copies of a square with opposite ends identified, with $g-1$ of these squares having

two 2-discs removed and the remaining 2 squares with one 2-disc removed, and the boundaries of these removed discs identified. This can be embedded in $\mathbb{R}^3$, and so inherits the subset topology. The atlas consists of local charts covering the surface where each domain of the chart is contractible on the surface. This is homeomorphic to a disc on $\mathbb{R}^2$, so this is a 2-manifold. $\square$

5 Crash Course in C^∞ Manifolds and Cohomology

Question 5.1. Let $\tau \in \mathbb{C}$ with $\text{Im}(\tau) > 0$. Define $E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, the quotient of $\mathbb{C}$ by the subgroup $\{m + n\tau \mid m, n \in \mathbb{Z}\}$. Show that E is a complex manifold.

Proof. Let $c_0, c_1 \in \mathbb{C}$ be a complex interval with $|\text{Re}(c_1) - \text{Re}(c_0)| < 1$ and $|\text{Im}(c_1) - \text{Im}(c_0)| < \text{Im}(\tau)$. Let $\pi : \mathbb{C} \rightarrow E$ be the map sending a complex number to its equivalence class. Then $\pi_{(c_0, c_1)}^{-1} : \pi(c_0, c_1) \rightarrow (c_0, c_1)$ are the charts of our atlas on E . (c_0, c_1) pairs with $|\text{Re}(c_1) - \text{Re}(c_0)| \geq 1$ or $|\text{Im}(c_1) - \text{Im}(c_0)| \geq \text{Im}(\tau)$ would give a noninjective map $\pi_{(c_0, c_1)}$. The transition functions are simply adding an integer to the real part and an integer multiple of $\text{Im}(\tau)$ to the complex part. We illustrate this with an example.

Let $\tau := a + bi$, where by definition $b > 0$. If we look at the two domains $V_0 := (0, \frac{2}{3} + \frac{2b}{3}i)$ and $V_1 := (2\frac{1}{3} + 2\frac{1}{3}bi, 3 + 3bi)$, we see that $\pi(V_0) \cap \pi(V_1) \neq \emptyset$, and

$$\begin{aligned}\pi(V_0) &= [0, \frac{2}{3} + \frac{2b}{3}i] \\ \pi(V_1) &= [\frac{1}{3} + \frac{b}{3}i, 1 + bi] \\ \Rightarrow \pi(V_0) \cap \pi(V_1) &= [\frac{1}{3} + \frac{b}{3}i, \frac{2}{3} + \frac{2b}{3}i]\end{aligned}$$

The image of this intersection is $[\frac{1}{3} + \frac{b}{3}i, \frac{2}{3} + \frac{2b}{3}i] \subset V_0$, and it is easy to see that its image under the transition function is $[2\frac{1}{3} + 2\frac{b}{3}i, 2\frac{2}{3} + 2\frac{2b}{3}i] \subset V_1$.

Since the transition functions are simply adding complex numbers, the transition functions are holomorphic. To make this explicit, the transition functions are given by

$$f(z) = \sum_{i=0}^{\infty} a_i(z - a)^i$$

where $\text{Re}(a)$ is an integer, $\text{Im}(a)$ is an integer multiple of $\text{Im}(\tau)$, and $\{a_i\}$ is the sequence $\{0, 1, 0, \dots\}$. $\square$

Question 5.2. Prove that if $\{v_1, \dots, v_k\}$ is a basis for a vector space V , then the set of vectors $\{v_{i_1} \wedge \dots \wedge v_{i_k} \mid 1 \leq i_1 < \dots < i_k \leq n\}$ is a basis for $\wedge^k V$.

Proof. Any nonzero vector in $\wedge^k V$ can be written as a sum of $v_{j_1} \wedge \dots \wedge v_{j_k}$ terms where all j_n are unique. If there is some $j_m = j_p$ in this expression, then we have

$$\begin{aligned} v_{j_1} \wedge \dots \wedge v_{j_m} \wedge v_{j_{m+1}} \dots \wedge v_{j_p} \wedge \dots \wedge v_{j_k} &= -v_{j_1} \wedge \dots \wedge v_{j_{m+1}} \wedge v_{j_m} \dots \wedge v_{j_p} \wedge v_{j_k} \\ &\vdots \\ &= \pm v_{j_1} \wedge \dots \wedge v_{j_{m+1}} \dots v_{j_m} \wedge v_{j_p} \wedge \dots \wedge v_{j_k} \\ &= \pm v_{j_1} \wedge \dots \wedge v_{j_{m+1}} \dots 0 \wedge \dots \wedge v_{j_k} \\ &= 0 \end{aligned}$$

Thus our k -tuple of j_n must be all unique. Since k is finite, we can rearrange the j_n in ascending order by flipping adjacent v_j s and adding a minus sign. Thus we have gotten to the prescribed basis.

It remains to show that each of these vectors are linearly independent. Suppose not. Then there exists three vectors such that

$$\dots av_i \wedge v_j \dots + \dots bv_k \wedge v_l \dots = \dots v_m \wedge v_n \dots$$

Then by the definition of the wedge product, we have $av_i + bv_k = v_m$ with $i \neq j \neq m$. This, however, contradicts the $\{v_i\}$ being a basis for V . Thus our vectors here must be linearly independent. $\square$

Question 5.3. Let $\{e_1, \dots, e_n\}$ be a basis for a vector space. Let $\{f_1, \dots, f_n\}$ be a change of basis with $f_i = \sum_j a_{ij} e_j$. Prove that

$$f_1 \wedge \dots \wedge f_n = \det(A) e_1 \wedge \dots \wedge e_n$$

where A is the matrix with entries a_{ij} .

Proof. We will use induction. Suppose $n = 2$. Then

$$\begin{aligned} f_1 \wedge f_2 &:= (a_{11}e_1 + a_{12}e_2) \wedge (a_{21}e_1 + a_{22}e_2) \\ &= (a_{11}a_{22} - a_{12}a_{21})e_1 \wedge e_2 \end{aligned}$$

Now suppose this is true for n . In an $(n + 1)$ -dimensional vector space, we have

$$\begin{aligned} f_1 \wedge \dots \wedge f_n \wedge f_{n+1} &:= \left(\sum_j^{n+1} a_{1j} e_j \right) \wedge \dots \wedge \left(\sum_j^{n+1} a_{nj} e_j \right) \wedge \left(\sum_j^{n+1} a_{(n+1)j} e_j \right) \\ &= \left(\sum_{i=1}^n \det(A_i) e_i \right) \wedge f_{n+1} \end{aligned}$$

where e_i is the wedge product $e_1 \wedge \dots \wedge e_n$ without e_i , and A_i is the corresponding change of basis determinant. Thus we have

$$\begin{aligned}
 f_1 \wedge \dots \wedge f_n \wedge f_{n+1} &= \left(\sum_{i=1}^n \det(A_i) e_i \right) \wedge f_{n+1} \\
 &= \left(\sum_{i=1}^n \det(A_i) e_i \right) \wedge \sum_{j=1}^{n+1} a_{(n+1)j} e_j \\
 &= [\det(A_1) a_{(n+1)1} - \det(A_2) a_{(n+1)2} + \dots] e_1 \wedge \dots \wedge e_{n+1} \\
 &= \det(A) e_1 \wedge \dots \wedge e_{n+1}
 \end{aligned}$$

□

Question 5.4. Let X be an oriented manifold with boundary. Show that the definition of the induced orientation on ∂X is independent of the choice of coordinates.

Proof. Suppose we have a point $x \in \partial X$ such that we have two charts $\phi_1 : U_1 \rightarrow \mathbb{R}^n, \phi_2 : U_2 \rightarrow \mathbb{R}^n$ where $x \in U_1 \cap U_2$. Let the coordinates of x be $(x_1, \dots, x_n)$ under the ϕ_1 coordinate chart, and $(y_1, \dots, y_n)$ under ϕ_2 . Since X is orientable, $\det\left(\frac{\partial y_j}{\partial x_i}\right) > 0$, so $(y_1, \dots, y_n)$ is consistent with X 's orientation if and only if $(x_1, \dots, x_n)$ is consistent with X 's orientation. If we now restrict our charts to ∂X

$$\phi_1|_{\partial X} : \partial X \rightarrow H_n, \phi_2|_{\partial X} : \partial X \rightarrow H_n$$

we have coordinates $(x_1, \dots, x_{n-1}, 0)$ and $(y_1, \dots, y_{n-1}, 0)$. By the definition given in the text, these coordinates are consistent with the orientation on ∂X if and only if $(x_1, \dots, x_n)$, and therefore $(y_1, \dots, y_n)$, are consistent with X 's orientation. Thus $\frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_{n-1}}$ and $\frac{\partial}{\partial y_1} \wedge \dots \wedge \frac{\partial}{\partial y_{n-1}}$ both lie in the same component of $\wedge^n T_x X - 0$. □

Question 5.5. Prove that if $H \subset G$ is a nonempty subset of G such that $h_1 - h_2 \in H$ whenever $h_1, h_2 \in H$, then H is a subgroup of G .

Proof. Because H is a subset of G and G is a group, all elements of H satisfy associativity. If h is in H , then $h - h = 0 \in H$, so H has G 's identity, and all elements in H have inverses. By definition, H is closed under our $+$ operation. □

Question 5.6. Prove that if G, H , and K are groups and $f : G \rightarrow H, g : H \rightarrow K$ are homomorphisms, then $g \circ f : G \rightarrow K$ is a homomorphism.

Proof. Consider two elements $g_1, g_2 \in G$. Then

$$g \circ f(g_1 + g_2) = g \circ (f(g_1) + f(g_2)) = g(f(g_1)) + g(f(g_2))$$

□

Question 5.7. Show that there is a category whose objects are groups and whose morphisms are group homomorphisms. Isomorphism in this category is just the usual notion of isomorphism of groups.

Proof. For any two groups, there exists at least the trivial homomorphism between them, that sends all elements in the preimage group to the identity. The identity morphism just sends all elements in a group to themselves. We proved above that for any three groups G, H, K , the composition of homomorphisms provide a map

$$\text{Mor}(G, H) \times \text{Mor}(H, K) \rightarrow \text{Mor}(G, K)$$

Since the identity morphisms don't change the elements in their groups, we have $f \circ 1_G = f = 1_H \circ f$ for all $f \in \text{Mor}(G, H)$. Furthermore, due to associativity of group elements and the definition of homomorphisms, we have associativity of homomorphisms. □

Question 5.8. Prove that $\partial_{k-1} \circ \partial_k = 0$.

Recall that

$$\partial_k : C_k(X) \rightarrow C_{k-1}(X), \quad \sum_r n_r f_r \mapsto \sum_{r,i} (-1)^i n_r (f_r \circ j_i)$$

where $f_i : \Delta_k \rightarrow X$ and $j_i : \Delta_{n-1} \rightarrow \Delta_n^i$ is defined by

$$j_i(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$$

Proof. For any $f : \Delta_k \rightarrow X$, we have

$$\begin{aligned} \partial_{k-1}(\partial_k(f)) &= \partial_{k-1}\left(\sum_i^k (-1)^i (f \circ j_i)\right) \\ &= \sum_m^{k-1} (-1)^m \left(\sum_i^k (-1)^i (f \circ j_i) \circ j_m\right) \\ &= \sum_m^{k-1} \sum_i^k (-1)^{m+i} (f \circ j_i) \circ j_m \end{aligned}$$

because ∂_k are homomorphisms. This sum contains two terms of equal simplices:

$$\begin{aligned} j_a \circ j_b(x_1, \dots, x_n) &= (x_1, \dots, x_{a-1}, x_{a+1}, \dots, x_{b-1}, x_{b+1}, \dots, x_n) \\ &= j_b \circ j_{a+1}(x_1, \dots, x_n) \end{aligned}$$

These are two terms of equal and opposite sign, as $(-1)^{a+b} = -(-1)^{a+1+b}$. Thus our sum amounts to 0 for every $f_r \in C_k(X)$. $\square$

Question 5.9. Prove that $\delta_{k+1} \circ \delta_k = 0$.

Proof. For $Z \in C_{k+2}(X)$ and $f \in C^k(X)$, we have

$$\begin{aligned} \delta_{k+1} \circ \delta_k(f)(Z) &= \delta_{k+1}(f(\partial_{k+1}(\cdot)))(Z) \\ &= f(\partial_{k+1}(\partial_{k+2}(Z))) \end{aligned}$$

In the previous problem we proved that $\partial_{k+1}(\partial_{k+2}(\cdot)) = 0$, so $\delta_{k+1} \circ \delta_k = 0$. $\square$

Question 5.10. Prove that the pairing

$$\langle \cdot, \cdot \rangle : H^k(X) \times H_k(X) \rightarrow \mathbb{Z}, \langle [f], [Z] \rangle = f(Z)$$

where $f \in Z^k(X)$ and $Z \in Z_k(X)$ are representatives of the cohomology class $[f]$ and the homology class $[Z]$, respectively, is well defined.

Proof. We need to check that there are no problems with the (co)homology elements representing their (co)homology classes. Thus we need to ensure that the pairing is zero if $[Z]$ is a boundary or if $[f]$ is a coboundary.

If the former is true, then $[Z] = \partial_{k+1} Z'$ for some $Z' \in C_{k+1}(X)$. Then the pairing is $f(\partial_{k+1}(Z')) = (\delta_k(f))(Z')$. This is equal to zero because f is in $Z^k(X) = \text{Ker}(\delta_k)$. Thus if $[Z]$ is a boundary, then the pairing is zero.

If $[f]$ is a coboundary, then there is some $f' \in C_{k-1}$ such that $f = \delta_{k-1}(f')$. Then our pairing is $\delta_{k-1}(f')(Z) = f'(\partial_k(Z))$. This is equal to zero because $Z \in Z_k(X) = \text{Ker}(\delta_k)$. Thus, and the pairing is well-defined. $\square$

Question 5.11. Show that any two hyperplanes in $\mathbb{P}^n$ represent the same homology class $H \in H_{2n-2}(\mathbb{P}^n)$. For each k , find a submanifold representing $H^k \in H_{2n-2k}(\mathbb{P}^n)$, where H^k denotes an iterated intersection product of H with itself.

Proof. Recall that a hyperplane in $\mathbb{P}^n$ is a degree 1 hypersurface. First we prove that a hyperplane in $\mathbb{P}^n$ is a $(2n - 2)$ -manifold. Denote our hyperplane by

$$a_0x_0 + \dots + a_ix_i + \dots + a_nx_n = 0$$

where without loss of generality at least a_i is nonzero. Then we have

$$x_i = -\frac{1}{a_i}(a_0x_0 + \dots + a_nx_n)$$

Our atlas is given by the atlas

$$\begin{aligned} \mathbb{R}^{2n-2} &\rightarrow \mathbb{C}^{n-1} \rightarrow \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n \\ (a_0, b_0, \dots, a_{n-1}, b_{n-1}) &\mapsto (a_0 + ib_0, \dots, a_{n-1} + ib_{n-1}) := (c_0, \dots, c_{n-1}) \\ &\mapsto (c_0, \dots, -\frac{1}{a_i}(a_0c_0 + \dots + a_nc_n), \dots, c_{n-1}) \end{aligned}$$

All of these maps are smooth, so hyperplanes are $(2n - 2)$ -dimensional submanifolds of $\mathbb{P}^n$. Notice that the images of these charts for different hyperplanes can be continuously mapped from one another:

$$(x_0, \dots, -\frac{1}{a_i}(a_0x_0 + \dots + a_nx_n), \dots, x_n) \mapsto (x_0, \dots, -\frac{1}{b_j}(b_0x_0 + \dots + b_nx_n), \dots, x_n)$$

This is a continuous map with a continuous inverse, i.e. any two hyperplanes are homeomorphic. Therefore the simplicial complexes are the same, and so they both represent the same homology class in $H_{2n-2}(\mathbb{P}^n)$.

For each k , define $K \subset \mathbb{P}^n$ to be the intersection of k hyperplanes: $(x_0 = 0) \cap (x_1 = 0) \cap \dots \cap (x_{k-1} = 0)$. The intersection between $(0, \dots, 0, x_{k-2}, \dots, x_n)$ and the hyperplane $x_{k-1} = 0$ can be given local coordinates where the k^{th} hyperplane's coordinates can be rearranged from $(x_0, \dots, x_{k-2}, 0, x_k, \dots, x_n) \mapsto (x_0, \dots, x_{n-1}, 0)$ by a change of coordinates, so these hyperplanes all intersect transversely. Therefore, K is a submanifold. Since this is the intersection of k hyperplanes, and we proved above that all hyperplanes represent the same homology class, the k is represented by the intersection product given by $[H] \times \dots \times [H] = [H \cap \dots \cap H] := H^k \in H_{2n-2k}(\mathbb{P}^n)$. $\square$

6 Cellular Decompositions and Line Bundles

Question 6.1. *Prove that any two hyperplanes in $\mathbb{P}^n$ represent the same homology class in $H_{2n-2}(\mathbb{P}^n)$.*

Proof. Let

$$\begin{aligned} a_0x_0 + \dots + a_nx_n &= 0, \\ b_0x_0 + \dots + b_nx_n &= 0 \end{aligned}$$

be any two hyperplanes in $\mathbb{P}^n$. Assuming without loss of generality that $a_n, b_n \neq 0$, define the map $f : \mathbb{P}^{n-1} \times [0, 1]$ by

$$f(x_0, \dots, x_{n-1}, t) = (L_0(t)x_0, \dots, L_{n-1}(t)x_{n-1}, \frac{L_0(t)x_0 + \dots + L_{n-1}(t)x_{n-1}}{-L_n(t)})$$

where $L_i(t) = (1 - t)a_i + tb_i$.

Let

$$\{f_i : \Delta_{2n-2} \rightarrow \mathbb{P}^{n-1}\}$$

be a triangulation of $\mathbb{P}^{n-1}$ and

$$\begin{aligned} F_i : \Delta_{2n-2} \times [0, 1] &\rightarrow \mathbb{P}^n \\ F_i(v, t) &= f(f_i(v), t) \end{aligned}$$

maps given in the text, respectively. Let

$$\{g_j : \Delta_{2n-1}^j \rightarrow \Delta_{2n-2} \times [0, 1]\}$$

be a triangulation of $\Delta_{2n-2} \times [0, 1]$. Then the restriction maps

$$\{F_i|_{\Delta_{2n-1}^j} : \Delta_{2n-1}^j \rightarrow \mathbb{P}^{n-1} \times [0, 1]\}$$

form a triangulation Γ of $\mathbb{P}^{n-1} \times [0, 1]$.

Since Γ is a triangulation, $\partial\Gamma$ is a sum of $(2n - 2)$ -chains, and each of these chains is described by a map $\Delta_{2n-2} \rightarrow \mathbb{P}^{n-1} \times [0, 1]$. All such chains except those on $\mathbb{P}^{n-1} \times \{0\} \cup \mathbb{P}^{n-1} \times \{1\}$ occurs twice, once with a plus sign and once with a minus sign, since each $(2n - 2)$ -dimensional face of the simplices occurs twice, once for each of the two simplices which border it on opposite sides. The chains that are on $\mathbb{P}^{n-1} \times \{0\} \cup \mathbb{P}^{n-1} \times \{1\}$ are given by triangulations of $\mathbb{P}^{n-1}$.

Taking the boundary map of the Γ chain, we have

$$\partial_{2n-1}\Gamma = Z_1 - Z_0,$$

where Z_0, Z_1 are $(2n - 2)$ -cycles representing the triangulations of our boundary $\mathbb{P}^{n-1}$ s (they are cycles because $\mathbb{P}^{n-1}$ is boundaryless. Since Z_0, Z_1 are cycles, they are the kernel of ∂_{2n-2} . Therefore we have

$$\begin{aligned} Z_0 + \partial_{2n-1}\Gamma &= Z_1 \Rightarrow \\ \frac{\ker(\partial_{2n-2})}{\text{Im}(\partial_{2n-1})} + 0 &= \frac{\ker(\partial_{2n-2})}{\text{Im}(\partial_{2n-1})} \Rightarrow \\ H_{2n-2}(Z_0) &= H_{2n-2}(Z_1) \end{aligned}$$

Alternatively, we could just take the f map defined at the beginning and state that this is a homotopy, so any two hyperplanes in $\mathbb{P}^n$ are homotopic, and since homology groups are homotopic invariant, they represent the same homology class. $\square$

Question 6.2. *Prove that any two distinct hyperplanes in $\mathbb{P}^n$ intersect transversely.*

Proof. Let

$$\begin{aligned} a_0x_0 + \dots + a_nx_n &= 0 \\ b_0x_0 + \dots + b_nx_n &= 0 \end{aligned}$$

be two distinct hyperplanes in $\mathbb{P}^n$. Without loss of generality, assume $a_0 \neq 0$. Then we the intersection of these two planes satisfying

$$(b_1 - \frac{b_0a_1}{a_0})x_1 + \dots + (b_n - \frac{b_0a_n}{a_0})x_n = 0$$

This is just a hyperplane in $\mathbb{P}^{n-1}$, which can be identified with $\mathbb{P}^{n-2}$. We can identify coordinates on this $\mathbb{P}^{n-2}$ by

$$(0, x_1, \dots, x_n, 0)$$

which satisfies this book's definition of transversality. $\square$

Question 6.3. *Show that the definition of degree given by $d(Z)$*

$$[Z] = d(Z)H^i \in H^{2i}(X)$$

for $Z \subset \mathbb{P}^n$ is a submanifold of codimension i and H the cohomology class of any hyperplane in $\mathbb{P}^2$ is consistent with the notion of the degree of a hypersurface.

Proof. A degree d hypersurface described by $F(x)$ can continuously deform into a union of d hyperplanes by the equations

$$G_t := tF(x) + (1-t) \prod_{i=1}^d l_i(x) = 0$$

where $t \in [0, 1]$ and the $l_i(x)$ are homogeneous linear forms. By the homotopy (deformation) argument above, $[Z(F)] = \sum_{i=1}^d [Z(l_i)] = dH \in H^2(\mathbb{P}^n)$.

Turning to cellular homology, let $F'(x)$ define a hyperplane. Since this is 1-dimensional, we can define the homotopy

$$G'_t := tF'(x) + (1-t)X_i$$

where X_i is the subspace defined by $(x_0, \dots, x_i, 0, \dots, 0)$. $[X_i]$ is the generator for $H^2(\mathbb{P}^n)$, so all hyperplanes have the same homology class $H \in H^2(\mathbb{P}^n)$.

Since singular homologies and cellular homologies are isomorphic, each dH in singular homology has a corresponding element in cellular homology, determined by this isomorphism. Thus, if Z is a codimension i hypersurface, its homology class $d(Z)H \in H^{2i}$ corresponds to the degree of Z . $\square$

Question 6.4. Suppose that Z is a complex submanifold of $\mathbb{P}^n$ of complex codimension i . Suppose that you can find a complex i -dimensional linear subspace of $\mathbb{P}^n$ intersecting Z transversally in a finite set S . Prove that the number of points in S is equal to the degree of Z .

Proof. Denote our i -dimensional linear subspace by $P \subset \mathbb{P}^n$. We take the intersection product

$$[P] \cdot [Z] = \sum_{\alpha, \beta} n_\alpha m_\beta [P_\alpha \cap Z_\beta]$$

Since by the assumption of the problem we have $P_\alpha \cap Z_\beta$ is a finite set of points, it remains to find n_α and m_β . We know that $[Z] = dH^i \in H^{2(n-i)}(\mathbb{P}^n)$, i.e. $[Z] = \sum_{\beta=1}^d [H]$ where H is a hyperplane in $\mathbb{P}^n$. Thus our indexing set β has d elements, and each n_α is 1.

Since P can be deformed into X_i , they have the same homology class, and therefore $[P]$ is the generator of the cellular homology group $H_{2i}(\mathbb{P}^n) \cong \mathbb{Z}$. Since cellular homologies are isomorphic with singular homologies, $[P]$ is also the generator of $H_{2i}(\mathbb{P}^n)$ in singular homology. Thus $[P] = 1 \in \mathbb{Z} \cong H_{2i}(\mathbb{P}^n)$, and our indexing set α is just $\{1\}$.

Since P and Z intersect transversally, we add up their dimensions, and get that $P_\alpha \cap Z_\beta$ is 0-dimensional. Furthermore, $Z_\beta(H)$ is a hyperplane, which is complex 1-dimensional, and, when transversally intersecting an i -dimensional linear subspace, intersects at a single point, oriented with a $+$ since P, Z are complex manifolds.

Therefore our sum $\sum_{\alpha, \beta} n_\alpha m_\beta [P_\alpha \cap Z_\beta]$ is a sum of positively oriented points, and since $n_\alpha = 1$, the number of such points is d . $\square$

Question 6.5. Find the degree of the common zero locus of the polynomials

$$x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_1x_3 - x_2^2$$

in $\mathbb{P}^3$.

Proof. By inspection, the intersection of the first two zero loci results in the zero locus of the third:

$$\begin{aligned} x_0x_3 &= x_1x_2 \\ \rightarrow \frac{x_1x_2}{x_3}x_2 &= x_1^2 \\ \rightarrow x_2^2 &= x_1x_3 \end{aligned}$$

Therefore the intersection is not transverse, and the hypersurfaces are not in general position. Isolating x_2 and x_3 , we get

$$(x_0, x_1, x_2, x_3) \rightarrow (x_0, x_1, \frac{x_1^2}{x_0}, \frac{x_1^3}{x_0^2})$$

This allows us to reparametrize coordinates:

$$(x_0, x_1) \mapsto (x_0^3, x_0^2x_1, x_0x_1^2, x_1^3)$$

which is a rational curve of degree 3. Thus the common zero locus of these three polynomials has degree 3. $\square$

Question 6.6. Prove that the only holomorphic section of $\mathcal{O}_{\mathbb{P}^1}(1)$ are those obtained by dehomogenizing homogeneous linear polynomials:

For $U_0 = \mathbb{P}^1 \setminus \{(0, 1)\}$ and $U_1 = \mathbb{P}^1 \setminus \{(1, 0)\}$ and inverse charts $\psi_i : \mathbb{C} \rightarrow U_i$ given by

$$\psi_0(x) = (1, x), \psi_1(w) = (w, 1)$$

where $x = \frac{x_1}{x_0}$ and $w = \frac{x_0}{x_1}$ we have

$$s_0 = a_0 + a_1 x, \quad s_1 = a_0 w + a_1$$

as dehomogenizations of linear polynomials.

Proof. Note that, for transition functions $g_{\alpha\beta} := \psi_\alpha \psi_\beta^{-1}$, we have $g_{\alpha\beta}(z) = z^{-1}$.

Suppose we have a set of functions $\{s'_0, s'_1\}$ such that $s'_1(w) = x^{-1}s'_0(x)$ and $s'_0(x) = w^{-1}s'_1(w)$ (where we've replaced $g_{10}(x)$, $g_{01}(w)$ with their images), i.e.

$$x_1 s'_1\left(\frac{x_0}{x_1}\right) = x_0 s'_0\left(\frac{x_1}{x_0}\right)$$

Our s'_i must be holomorphic by definition, so locally we have

$$s'_i\left(\frac{x_0}{x_1}\right) = \sum_{i=0}^{\infty} a_i \left(\frac{x_0}{x_1} - a\right)^i$$

In order for $x_1 s'_1\left(\frac{x_0}{x_1}\right)$ to equal $x_0 s'_0\left(\frac{x_1}{x_0}\right)$ for all $(x_0, x_1) \in U_0 \cap U_1$, we can only have a_0, a_1 be nonzero in our convergent series, as the powers of the x_0 and x_1 will be mismatched. This yields s_0, s_1 defined above, so these are the only holomorphic sections of $\mathcal{O}_{\mathbb{P}^1}(1)$. $\square$

Question 6.7. Let $\{U_0, \dots, U_n\}$ be the standard open cover of $\mathbb{P}^n$. Let d be an integer.

- a Show that the functions $g_{\alpha\beta} = (x_\beta/x_\alpha)^d$ are the transition functions of a line bundle on $\mathbb{P}^n$, denoted $\mathcal{O}_{\mathbb{P}^n}(d)$.
- b If $d > 0$, show that this bundle coincides with the bundle $\mathcal{O}_{\mathbb{P}^n}(d)$ defined by $\mathcal{O}_{\mathbb{P}^n}(1)^{\otimes d}$.
- c If $d > 0$, show that the homogeneous polynomials of degree d on $\mathbb{P}^n$ naturally correspond via dehomogenization to the holomorphic sections of $\mathcal{O}_{\mathbb{P}^n}(d)$.

Proof. a The $\{U_\alpha\}$ are defined such that $x_\alpha \neq 0$ on U_α , so the function $g_{\alpha\beta} = \left(\frac{x_\beta}{x_\alpha}\right)^d$ is everywhere defined on U_α . Furthermore, $g_{\alpha\alpha} = \left(\frac{x_\alpha}{x_\alpha}\right)^d = 1$ and $g_{\alpha\beta}g_{\beta\gamma} = \left(\frac{x_\beta}{x_\alpha}\right)^d \left(\frac{x_\gamma}{x_\beta}\right)^d = \left(\frac{x_\gamma}{x_\alpha}\right)^d = g_{\alpha\gamma}$, satisfying the requirements to be transition functions of a line bundle on $\mathbb{P}^n$.

- b The line bundle $L \otimes L'$ is defined by multiplying the transition functions: $\{g_{\alpha\beta}g'_{\alpha\beta}\}$, so the line bundle $L^{\otimes d}$ of $\mathcal{O}_{\mathbb{P}^n}(1)^{\otimes d}$ is given by $g_{\alpha\beta}^d$, which is $(\frac{x_\beta}{x_\alpha})^d$.
- c We have $s_\alpha = g_{\alpha\beta}s_\beta$ in each $U_\alpha \cap U_\beta$. We must have $s_\alpha(x_\alpha, x_\beta)$ be polynomials of degree d in order to have the powers of x_α and x_β to match between s_α and s_β , by the same argument used above. If

$$s_\alpha(x_\alpha, x_\beta) = a_0 + a_1 \frac{x_\beta}{x_\alpha} + \dots + a_{d+1} \left(\frac{x_\beta}{x_\alpha}\right)^d,$$

i.e. degree d polynomials of $\frac{x_\beta}{x_\alpha}$, where $x_\alpha, x_\beta \in U_\alpha \cap U_\beta$, we have

$$\begin{aligned} g_{\alpha\beta}s_\beta &= \left(\frac{x_\beta}{x_\alpha}\right)^d \left(a_0 + a_1 \frac{x_\alpha}{x_\beta} + \dots + a_{d+1} \left(\frac{x_\alpha}{x_\beta}\right)^d\right) \\ &= a_0 \left(\frac{x_\beta}{x_\alpha}\right)^d + a_1 \left(\frac{x_\beta}{x_\alpha}\right)^{d-1} + \dots + a_{d+1} \\ &= s_\alpha \end{aligned}$$

which satisfies the holomorphic section requirement. Thus the holomorphic sections of $\mathcal{O}_{\mathbb{P}^n}(d)$ are dehomogenized polynomials of degree d . □

7 Enumerative Geometry of Lines

Question 7.1. Complete the details of the description of the cellular decomposition of $G(2, 4)$, describing all the cells.

Proof. From the text, the cellular decomposition is

$$L \subset \sigma_{2,1}(p, P) \subset \sigma_2(p) \cup \sigma_{1,1}(P) \subset \sigma_1(L) \subset G,$$

where

$$\begin{aligned} \sigma_1(L) &= \{l \in G \mid l \cap L \neq \emptyset\} \\ \sigma_2(p) &= \{l \in G \mid p \in l\}, \quad p = (0, 0, 0, 1) \\ \sigma_{1,1}(P) &= \{l \in G \mid l \subset P\}, \quad P = Z(x_0) \\ \sigma_{2,1}(p, P) &= \{l \in G \mid p \in l \subset P\} \end{aligned}$$

where L is the line defined by $x_0 = x_1 = 0$. ($Z(x_0)$ is the subspace defined by $x_0 = 0$.) The text also gives the 4-cell $G - \sigma_1(L)$ as the row space of

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix} \cong \mathbb{C}^4$$

and the 3-cell $\sigma_1(L) - \sigma_2(p) - \sigma_{1,1}(P)$ is given by

$$\begin{pmatrix} 1 & a & 0 & b \\ 0 & 0 & 1 & c \end{pmatrix} \cong \mathbb{C}^3$$

The 2-cell is given by $(\sigma_2(p) \cup \sigma_{1,1}(P)) - \sigma_{2,1}(p, P)$, i.e. the set of lines l that either contain p while not being in $Z(x_0)$ plus the set of lines that don't contain p and are contained in $Z(x_0)$. For the former, we need the 2-dimensional subspace generated by two vectors

$$e_1 := \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{pmatrix}, e_2 := \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix}$$

To ensure that $\alpha e_1 + \beta e_2 \notin P$ spanning α and β , we must have either a_0 or a_1 equal to 1. We will set $a_0 = 1$. To ensure that $p \in l$ for at least one combination of α, β , we must have $\alpha = 0$, so $e_1 = p$. This covers all values in the fourth entry of the resulting vector, so we can set $d_0 = 0$. The remaining b_0, c_0 can be any value and still fit these criteria, so this space is given by the row space of

$$\begin{pmatrix} 1 & b_0 & c_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for the set $\{l \in G | p \notin l, l \subset P\}$, we note that both a_0 and a_1 in the above definitions of e_1, e_2 must be 0 to ensure any resulting vector from spanning α, β remains in P . Furthermore, to ensure the $p \notin l$ for any α, β requires $d_0 = d_1 = 0$. The remaining second and third entries can be any values, so we set $c_0 = 1$ and $b_1 = 1$, giving the row space

$$\begin{pmatrix} 0 & b_0 & 1 & 0 \\ 0 & 1 & c_1 & 0 \end{pmatrix}$$

Both of these row spaces are disjoint and isomorphic to $\mathbb{C}^2$, thus these are the 2-cells.

The 1-cell given by $\sigma_{2,1}(p, P) - \{L\}$, we must first have $a_0 = a_1 = 0$ to remain in P . We must also have either b_0 or b_1 be 1 to ensure we stray from L , so we'll set $b_0 = 1$. To ensure p is hit, we will set $b_1 = c_1 = 0$ and $d_1 = 1$. The third entry in the resulting vector can be anything, so we have the resulting vector given by $(0, \alpha, \alpha c_0, \beta)$ with α, β spanning. Thus we have the 1-cell given by the row space of

$$\begin{pmatrix} 0 & 1 & c_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is in bijection with $\mathbb{C}$, and completes our description of the cells in our cellular decomposition of G . $\square$

Question 7.2. Let $p = (1, 0, 0, 0)$ and $p' = (0, 1, 0, 0)$ in $\mathbb{P}^3$. Show explicitly that $\sigma_2(p)$ and $\sigma_2(p')$ intersect transversely by using the local coordinates (a, b, c, d) to describe the row space of

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$$

near the line L joining p and p' . Note that L itself corresponds to the origin $(0, 0, 0, 0)$. Find the equations in (a, b, c, d) defining $\sigma_2(p)$ and $\sigma_2(p')$, and note that these satisfy the condition for a transverse intersection.

Proof. In computing $\sigma_2(p)$, in order to ensure $p \in V$, where V is the row space of

the above matrix, we need $a = b = 0$. When this is the case, $\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ c \\ d \end{pmatrix}$ hits p

when $\alpha = 1, \beta = 0$. Outside of that, c and d can take any value, so $\sigma_2(p)$ is given by the 2-dimensional subspace of $(a, b, c, d) \cong \mathbb{C}^4$ given by $(0, 0, c, d) \cong \mathbb{C}^2$.

Similarly, for $\sigma_2(p')$, we must have $c = d = 0$ and a, b can take any values, so so $\sigma_2(p')$ is given by the 2-dimensional subspace of $(a, b, c, d) \cong \mathbb{C}^4$ given by $(a, b, 0, 0) \cong \mathbb{C}^2$. This is the exact definition of transversality given in the text. $\square$

Question 7.3. Verify the multiplication table for $H^*(G(2,4))$.

$\cdot$	1	σ_1	σ_2	$\sigma_{1,1}$	$\sigma_{2,1}$	$\sigma_{2,2}$
1	1	σ_1	σ_2	$\sigma_{1,1}$	$\sigma_{2,1}$	$\sigma_{2,2}$
σ_1	σ_1	$\sigma_2 + \sigma_{1,1}$	$\sigma_{2,1}$	$\sigma_{2,1}$	$\sigma_{2,2}$	0
σ_2	σ_2	$\sigma_{2,1}$	$\sigma_{2,2}$	0	0	0
$\sigma_{1,1}$	$\sigma_{1,1}$	$\sigma_{2,1}$	0	$\sigma_{2,2}$	0	0
$\sigma_{2,1}$	$\sigma_{2,1}$	$\sigma_{2,2}$	0	0	0	0
$\sigma_{2,2}$	$\sigma_{2,2}$	0	0	0	0	0

Proof. The first row is trivial, because the intersection of subspaces like e.g. ps , Ps , and Ls with G itself acts by the identity on the respective subspaces.

Due to commutativity of intersection and multiplication of integers in $\mathbb{Z}$, the table is symmetric.

$\sigma_1 \cdot \sigma_1$ is proven in the text to be $\sigma_2 + \sigma_{1,1}$.

$\sigma_1 \cdot \sigma_2$ is given by $\{l \in G | p \in l, l \cap L \neq \emptyset\}$, i.e. the set of lines that intersect a point and a line. In other words, we want to find, for a point $p = (a', b', c', d')$ in general position, which a, b, c, d yield

$$\alpha \begin{pmatrix} 1 \\ a \\ 0 \\ b \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \\ c \end{pmatrix} = \begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix}$$

This gives the constraints $a = \frac{b'}{a'}$ and $b = \frac{d' - c'c}{a'}$. This means that, without loss of generality c can take any value and still give a line that intersects p and L . The lines must therefore be contained on a plane given by (a, c) . To intersect with p , we have a constraint on a as well. Thus we have a set of lines on a plane that intersect p , i.e. $\sigma_{2,1} = \{l \in G | p \in l \subset P\}$.

For $\sigma_1 \cdot \sigma_{1,1}$, we seek $\{l \in G | l \cap L \neq \emptyset, l \subset P\}$. In a transversal intersection, a line L intersects a plane P in a point. Thus lines in P intersecting with L must contain this point. Therefore $\sigma_1 \cdot \sigma_{1,1} = \{l \in G | p \in l \subset P\} = \sigma_{2,1}$.

For $\sigma_1 \cdot \sigma_{2,1}$, we seek $\{l \in G | p \in l \subset P, l \cap L \neq \emptyset\}$. From the previous reasoning, a line contained in a plane P that intersects with L must intersect with the single point of intersection $L \cap P$. To also intersect with another point $p \in P$ reduces our set to a single line, as only one line intersects two points. Thus our set is $\{L\} = \sigma_{2,2}$.

For $\sigma_1 \cdot \sigma_{2,2}$, we need the intersection of a line with another line. In general position, however, this is the empty set, so $\sigma_1 \cdot \sigma_{2,2} = 0$.

σ_2^2 is shown to be $\sigma_{2,2}$ and $\sigma_2 \cdot \sigma_{1,1}$ is shown to be 0 in the text.

$\sigma_2 \cdot \sigma_{2,2}$ is the intersection of a line with a point, which in general position is the empty set, so $\sigma_1 \cdot \sigma_{2,2} = 0$.

$\sigma_{1,1}^2$ is given by $\sigma_{2,2}$ in the text.

$\sigma_{1,1} \cdot \sigma_{2,1}$ is $\{l \in G | l \subset P, p \in l \subset P'\}$. The intersection $P \cap P'$ is a line, and this line does not intersect a point in general position, so $\sigma_{1,1} \cdot \sigma_{2,1} = 0$.

$\sigma_{1,1} \cdot \sigma_{2,2}$ is the intersection of lines in a plane with another line in general position. Since a line in general position isn't in a plane, the intersection is 0.

$\sigma_{2,1}^2$ is given by $\{l \in G | p, p' \in l, l \subset P, P'\}$. Any l that must exist in the intersection of two planes in general position must be a single line. This line will not intersect both p and p' , as both points will in general be off of this line, so $\sigma_{2,1}^2 = 0$.

$\sigma_{2,1} \cdot \sigma_{2,2}$ requires a selected line to contain a point and be contained in a plane, both in general position. A general line won't be contained in a general plane, so $\sigma_{2,1} \cdot \sigma_{2,2} = 0$.

$\sigma_{2,2}^2$ is asking for a line that is two distinct lines (since we're in general position), which can't happen, so $\sigma_{2,2}^2 = 0$. $\square$

Question 7.4. *a Let $C \subset \mathbb{P}^3$ be a curve of degree d . Let Z be the set of lines in $\mathbb{P}^3$ which intersect C . Find the cohomology class $[Z] \in H^*(G(2, 4))$.*

b How many lines in $\mathbb{P}^3$ meet each of four given general curves C_i of respective degrees d_i ?

Proof. a Z must be the cohomology class $[\{l \in \mathbb{P}^3 | l \cap C \neq \emptyset\}]$. This is

the same as the cohomology given by $[\bigcap_{i=1}^d \{l \in \mathbb{P}^3 | l \cap L_i \neq \emptyset\}]$ since our

hypersurface C is homotopy equivalent to $\prod_{i=1}^d L_i$. This is just $\bigcap_{i=1}^d \sigma_1(L_i) =$

$\sum_{i=1}^d \sigma_1$. Therefore $[Z] = d\sigma_1 \in H^2(G(2, 4))$.

b We look for the intersection of these four cohomology classes:

$$\begin{aligned}
d_1\sigma_1 \cap d_2\sigma_1 \cap d_3\sigma_1 \cap d_4\sigma_1 &= \prod_{i=1}^4 d_i\sigma_1^4 \\
&= \prod_{i=1}^4 d_i(\sigma_2 + \sigma_{1,1})^2 \\
&= \prod_{i=1}^4 d_i(\sigma_2^2 + 2\sigma_2\sigma_{1,1} + \sigma_{1,1}^2) \\
&= \prod_{i=1}^4 d_i 2\sigma_{2,2}
\end{aligned}$$

Thus we have $\int_G \prod_{i=1}^4 d_i 2\sigma_{2,2} = 2d_1d_2d_3d_4 \int_G \sigma_{2,2} = 2d_1d_2d_3d_4$ lines in $\mathbb{P}^3$ meeting all four hypersurfaces.

□

Question 7.5. *How many lines in $\mathbb{P}^4$ intersect each of six general 2-planes? How many lines in $\mathbb{P}^4$ intersect each of three general 2-planes and pass through a fixed point?*

Proof. The set of lines that intersect a single 2-plane in general position in $\mathbb{P}^4$ is given by the Schubert cycle

$$\sigma_{a,b} = \{l \in G(2,5) \mid l \cap \mathbb{P}(V_{4-a}) \neq \emptyset, l \subset \mathbb{P}(V_{4+1-b})\}$$

for $a = 1, b = 0$. (This is because a 2-plane is 3-dimensional, i.e. $\mathbb{P}(V_3)$, and no further restrictions on l , so we set $b = 0$ to ensure $l \subset \mathbb{P}(V_5) = \mathbb{P}^4$.) Therefore we seek the value of $\sigma_{1,0}^6$. To compute this, we'll use Pieri's Formula given in the book, since I'm feeling lazy right now:

$$\sigma_a \cdot \sigma_{b,c} = \sum_{\{(d,e) \mid d+e=a+b+c, n-1 \geq d \geq b \geq e \geq c\}} \sigma_{d,e}$$

We then have

$$\begin{aligned}
\sigma_{1,0}^2 &= \sigma_{2,0} + \sigma_{1,1} \\
\sigma_{2,0} \cdot \sigma_{1,0} &= \sigma_{3,0} + \sigma_{2,1} \\
\sigma_{1,0} \cdot \sigma_{1,1} &= \sigma_{2,1} \\
\sigma_{3,0} \cdot \sigma_{1,0} &= \sigma_{4,0} + \sigma_{3,1} \\
\sigma_{1,0} \cdot \sigma_{2,1} &= \sigma_{3,1} + \sigma_{2,2} \\
\sigma_{4,0} \cdot \sigma_{1,0} &= 0 \\
\sigma_{1,0} \cdot \sigma_{3,1} &= \sigma_{3,2} \\
\sigma_{1,0} \cdot \sigma_{2,2} &= \sigma_{3,2} \\
\sigma_{1,0} \cdot \sigma_{3,2} &= \sigma_{3,3}
\end{aligned}$$

Thus we have

$$\begin{aligned}
\sigma_{1,0}^6 &= \sigma_{1,0}^2 \cdot \sigma_{1,0}^4 \\
&= (\sigma_{2,0} + \sigma_{1,1}) \cdot \sigma_{1,0}^4 \\
&= (\sigma_{3,0} + 2\sigma_{2,1}) \cdot \sigma_{1,0}^3 \\
&= (\sigma_{4,0} + 3\sigma_{3,1} + 2\sigma_{2,2}) \cdot \sigma_{1,0}^2 \\
&= 5\sigma_{3,2} \cdot \sigma_{1,0} \\
&= 5\sigma_{3,3}
\end{aligned}$$

Following the

$$\langle \sigma_{a,b}, \sigma_{c,d} \rangle = \begin{cases} 1 & c = n - 1 - b, \ d = n - 1 - a \\ 0 & \text{otherwise} \end{cases}$$

pairing rule given in the text, $\sigma_{3,3}$'s dual is $\sigma_{0,0} = 1$. We then have

$$\int_{G(2,5)} \sigma_{1,0}^6 = \int_{G(2,5)} 5\sigma_{3,3} = 5 \int_{G(2,5)} \sigma_{3,3} = 5\langle \sigma_{3,3}, \sigma_{0,0} \rangle = 5,$$

giving us five lines in $\mathbb{P}^4$ intersecting six general 2-planes.

For the second question, from the same reasoning above, we seek $\int_{G(2,5)} \sigma_{1,0}^3 \cdot \sigma_{3,0}$. The computations above give us

$$\begin{aligned}
\sigma_{1,0}^3 \cdot \sigma_{3,0} &= (\sigma_{3,0} + 2\sigma_{2,1}) \cdot \sigma_{3,0} \\
&= \sigma_{3,0}^2 + 2\sigma_{3,0} \cdot \sigma_{2,1}
\end{aligned}$$

Pieri's Formula gives us

$$\begin{aligned}\sigma_{3,0}^2 &= \sigma_{3,3} \\ \sigma_{3,0} \cdot \sigma_{2,1} &= 0\end{aligned}$$

So we have

$$\int_{G(2,4)} \sigma_{1,0}^3 \cdot \sigma_{3,0} = \int_{G(2,4)} \sigma_{3,3} = 1$$

Thus there is one line in $\mathbb{P}^4$ intersecting three general 2-planes passing through a fixed point. $\square$

Question 7.6. *Show that*

$$(p, v_\alpha) \sim (q, v_\beta) \text{ if } p = q \text{ and } v_\alpha = g_{\alpha\beta}(p) \cdot v_\beta$$

(where $(p, v_\alpha) \in U_\alpha \times \mathbb{C}^r$ and $(q, v_\beta) \in U_\beta \times \mathbb{C}^r$, given transition functions $g_{\alpha\beta}$ on X a manifold with atlas sets $\{U_\alpha\}$) is an equivalence relation and that $V(E)$ is a complex manifold of dimension $\dim X + r$.

Proof. We start with reflexivity: since $g_{\alpha\beta}$ are transition functions, we have $g_{\alpha\alpha} = 1$. Therefore $(p, v_\alpha) \sim (p, v_\alpha)$.

For symmetry, a requirement of transition functions for a rank r vector bundle subordinate to the cover $\{U_\alpha\}$ is that the $g_{\alpha\beta}$ transition functions are invertible. Therefore, if $(p, v_\alpha) \sim (q, v_\beta)$, then we have $p = q$ and a transition function $g_{\alpha\beta}$ such that $v_\alpha = g_{\alpha\beta}v_\beta$, we have another transition function $g_{\alpha\beta}^{-1}$ such that $v_\beta = g_{\alpha\beta}^{-1}v_\alpha$. Therefore we have $(q, v_\beta) \sim (p, v_\alpha)$.

For transitivity, a further requirement of transition functions for a rank r vector bundle subordinate to the cover $\{U_\alpha\}$ is that $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$ on each nonempty $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. Therefore, if we have $(p, v_\alpha) \sim (q, v_\beta)$ and $(q, v_\beta) \sim (m, v_\gamma)$, then we have $p = q = m$ and transition functions $g_{\alpha\beta}$ and $g_{\beta\gamma}$ such that $v_\alpha = g_{\alpha\beta}v_\beta$ and $v_\beta = g_{\beta\gamma}v_\gamma$. Since $p = q = m$, we must have nonempty $U_\alpha \cap U_\beta \cap U_\gamma$, and therefore a transition function $g_{\alpha\gamma} = g_{\alpha\beta}g_{\beta\gamma}$. We then have

$$g_{\alpha\gamma}v_\gamma = g_{\alpha\beta}g_{\beta\gamma}v_\gamma = g_{\alpha\beta}v_\beta = v_\alpha$$

so $(p, v_\alpha) \sim (m, v_\gamma)$.

$V(E)$ is defined as the quotient of the disjoint union of the $U_\alpha \times \mathbb{C}^r$ by this equivalence relation. Since X is a complex topological manifold, there are charts $\phi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{C}^{\dim(X)}$. Therefore we define charts on $V(E)$ via

$$\phi_\alpha \times 1 : U_\alpha \times \mathbb{C}^r \rightarrow V_\alpha \times \mathbb{C}^r \subset \mathbb{C}^{\dim(X)} \times \mathbb{C}^r \cong \mathbb{C}^{\dim(X)+r}$$

where 1 denotes the $r \times r$ identity matrix on $\mathbb{C}^r$. Quotienting the $U_\alpha \times \mathbb{C}^r$ by the equivalence relation, we check well-definedness: Suppose $(p, v_\alpha) \sim (q, v_\beta)$. Then

$$\begin{aligned}\phi_\alpha \times 1(p, v_\alpha) &= \phi_\alpha(p) \times v_\alpha = \phi_\alpha(p) \times g_{\alpha\beta} v_\beta, \\ \phi_\beta \times 1(q, v_\beta) &= \phi_\beta(q) \times v_\beta = \phi_\beta(q) \times v_\beta = \phi_\alpha(p) \times v_\beta\end{aligned}$$

So the transition maps are given by $1_X \times g_{\alpha\beta}$, which are holomorphic. $\square$

Question 7.7. *a Try to make precise the meaning of the statement “the fiber of π over x has a vector space structure, the operations depending holomorphically on x ” in the geometric construction of $V(E)$.*

- b Prove that the correspondence between vector bundles and geometric vector bundles is a one-to-one correspondence.*
- c Finally, prove that under the above correspondence, holomorphic sections are in one-to-one correspondence with holomorphic maps $s : X \rightarrow V(E)$ such that $\pi \circ s = 1_X$.*

Proof. *a* In the construction of the geometric vector bundle in the text, we are given a holomorphic mapping $\pi : V \rightarrow X$ with a covering of $\{U_\alpha\}$ of X such that $\pi^{-1}(U_\alpha) \simeq U_\alpha \times \mathbb{C}^r$. Let $i_\alpha : U_\alpha \times \mathbb{C}^r \rightarrow \pi^{-1}(U_\alpha)$ be the isomorphism of this neighborhood. Fixing $x \in U_\alpha$, we define the isomorphism $j_\alpha(x) = i_\alpha(x, v)$. We define the vector space operations on $\pi^{-1}(x)$ by

$$\begin{aligned}v + w &= j_\alpha(j_\alpha^{-1}(v) + j_\alpha^{-1}(w)) \\ cv &= j_\alpha(cj_\alpha^{-1}(v))\end{aligned}$$

Because π is holomorphic, j_α varies holomorphically depending on x , so the vector space operations depend holomorphically on x .

- b* Suppose we have two rank- r vector bundles V, V' corresponding to the same geometric vector bundle $V(E)$. Then V can be trivialized over a cover $\{U_\alpha\}$ of X , and V' can be trivialized over another cover $\{U'_\alpha\}$ of X , such that there exist isomorphisms $\pi : V \rightarrow X, \pi' : V' \rightarrow X$, such that $\pi^{-1}(U_\alpha) \simeq U_\alpha \times \mathbb{C}^r$ and $\pi'^{-1}(U'_\alpha) \simeq U'_\alpha \times \mathbb{C}^r$. Since V, V' correspond to $V(E)$, $V(E)$ is the quotient of both of the equivalence relations

$$\begin{aligned}(p, v_\alpha) &\sim (q, v_\beta), \quad v_\alpha = g_{\alpha\beta} v_\beta \\ (p, v'_\alpha) &\sim (q, v'_\beta), \quad v'_\alpha = g'_{\alpha\beta} v'_\beta\end{aligned}$$

Let $p \in U_\alpha \cap U_\beta$ in the V bundle with $p \in U'_\alpha \cap U'_\beta$ in the V' bundle, and $\{e_i\}$ be a basis of $\pi^{-1}(U_\alpha)$ and $\{e'_i\}$ be a basis of $\pi'^{-1}(U'_\alpha)$. We use these to construct a linear map f_α that sends each $e_i \mapsto e'_i$. Since the vector space operations on a fiber depends holomorphically on the fiber's point, this linear map also depends holomorphically on the fiber's point, and this sends v_α to v'_α . We do the same for the bases of $\pi^{-1}(U_\beta)$ and $\pi'^{-1}(U'_\beta)$, and call the resulting map f_β , which sends v_β to v'_β . Therefore we have

$$\begin{aligned} g'_{\alpha\beta} f_\beta v_\beta &= f_\alpha g_{\alpha\beta} v_\beta \\ \Rightarrow g'_{\alpha\beta} v_\beta &= f_\alpha g_{\alpha\beta} f_\beta^{-1} v_\beta \\ \Rightarrow g'_{\alpha\beta} &= f_\alpha g_{\alpha\beta} f_\beta^{-1} \end{aligned}$$

on $\{U_\alpha \cap U_\beta\}$, which is the equivalence given in the text. Therefore V is isomorphic to V' , and the correspondence between vector bundles and geometric vector bundles is one-to-one.

- c Suppose we have two holomorphic sections s, s' with bundle trivializations $\{U_\alpha\}, \{U'_\alpha\}$ corresponding to a holomorphic map $s_{geo} : X \rightarrow V(E)$ such that $\pi \circ s_{geo} = 1_X$. Then restricting s_{geo} to covers $U_\alpha \in \{U_\alpha\}$ and $U'_\alpha \in \{U'_\alpha\}$, we identify

$$\pi^{-1}(U_\alpha) \simeq U_\alpha \times \mathbb{C}^r, \quad \pi'^{-1}(U'_\alpha) \simeq U'_\alpha \times \mathbb{C}^r$$

and restrict $s_{geo}|_{U_\alpha}$ and $s_{geo}|_{U'_\alpha}$ as $x \mapsto (x, s_{geo,\alpha}(x))$ and $x \mapsto (x, s'_{geo,\alpha}(x))$, respectively, where $s_{geo,\alpha}, s'_{geo,\alpha}$ are vector-valued holomorphic functions. This is well defined, because if $x \in U_\alpha \cap U_\beta$ and $x \in U'_\alpha \cap U'_\beta$,

$$(x, s_{geo,\beta}(x)) = (x, g_{\alpha\beta} s_{geo,\alpha}(x)), \quad (x, s'_{geo,\beta}(x)) = (x, g'_{\alpha\beta} s'_{geo,\alpha}(x))$$

which are identified under the vector bundle equivalence relation. This is the case because the vector bundle sections correspond to s_{geo} . Under the above correspondence, we can write linear maps $\{f_\alpha\}$ that send $s_{geo,\alpha}$ to $s'_{geo,\alpha}$. These give equivalences between transition functions $g_{\alpha\beta} \sim g'_{\alpha\beta}$:

$$\begin{aligned} (x, s'_{geo,\beta}(x)) &= (x, f_\beta s_{geo,\beta}(x)) = (x, f_\beta g_{\alpha\beta} s_{geo,\alpha}(x)) \\ &= (x, f_\beta g_{\alpha\beta} f_\alpha^{-1} s'_{geo,\alpha}(x)) \\ &= (x, g'_{\alpha\beta} s'_{geo,\alpha}(x)) \\ \Rightarrow f_\beta g_{\alpha\beta} f_\alpha^{-1} &= g'_{\alpha\beta} \end{aligned}$$

Since the transition functions are identified, the sections s, s' are identified, and therefore the holomorphic sections are in one-to-one correspondence with holomorphic maps $s : X \rightarrow V(E), \pi \circ s = 1_X$. $\square$

Question 7.8. Let $Q = \bigcup_{l \in G(2, n+1)} \{l\} \times Q_l$ be the disjoint union of the $\{l\} \times Q_l$. (Q_l is the 2-dimensional vector space of homogeneous linear polynomials on l .) Show that Q has the structure of a rank 2 vector bundle on $G(2, n+1)$ by giving an explicit trivialization.

Proof. Let $\{x_0, \dots, x_n\}$ be a basis of $\mathbb{C}^{n+1}$. Let U_{x_i, x_j} be the subset of $G(2, n+1)$ consisting of lines where their spans can be written as two vectors where the i^{th} entries are 1 and 0 and the j^{th} entries are 0 and 1, respectively, with the requirement that $i < j$. For example, U_{x_0, x_1} consists of the lines l for which their respective spans can be given by elements of the row space

$$\begin{pmatrix} 1 & 0 & a_{11} & \dots & a_{1(n-1)} \\ 0 & 1 & a_{21} & \dots & a_{2(n-1)} \end{pmatrix}$$

The trivialization then gives $Span(x_i, x_j) \cong \mathbb{C}^2$ over each U_{x_i, x_j} . These U_{x_i, x_j} clearly span $G(2, n+1)$: If there is an l in $G(2, n+1)$, let it be spanned by two basis vectors $v = (v_0, \dots, v_n), w = (w_0, \dots, w_n)$. Row reducing the matrix $\begin{pmatrix} v \\ w \end{pmatrix}$ into Echelon form gives a matrix of the form U_{x_0, x_i} . Transition functions are simply given by the 2D identity matrix, mapping $x_{i_1} \mapsto x_{i_2}$ by a 1 in the upper left corner and $x_{j_1} \mapsto x_{j_2}$ by a 1 in the lower right corner, and 0 elsewhere. Under the identification $(l, ax_{i_1} + bx_{j_1}) \sim (l', g_{x_{i_1}x_{j_1}, x_{i_2}x_{j_2}}(ax_{i_2} + bx_{j_2}))$, this gives the structure of a rank-2 vector bundle. $\square$

Question 7.9. Find the number of lines in $\mathbb{P}^3$ which lie in a general degree 2 hypersurface and meet a given line.

Proof. Let Q be the rank-2 fiber bundle of $G(2, 4)$ defined above. A line l in a degree 2 hypersurface X , where $X = Z(F(x))$ for F our defining degree 2 polynomial. Then $l \in X$ when $F|_l = 0$. As in the text, F is a section of Q 's second symmetric power $Sym^2(Q)$. This is a rank-3 bundle, generated by $\{x_i^2, x_i x_j, x_j^2\}$ over $U_{x_i x_j}$. Thus we seek to find

$$\int_{G(2,4)} c_3(Sym^2(Q)) \cap \sigma_1$$

Using the splitting principle, we assume that $Q \cong L_1 \oplus L_2$ is a direct sum of line bundles, with first Chern classes denoted $\alpha = c_1(L_1), \beta = c_1(L_2)$, with $c(Q) = (1 + \alpha)(1 + \beta)$. We are given in the text that $c(Q) = 1 + \sigma_1 + \sigma_{1,1}$, so

$$\alpha + \beta = \sigma_1, \alpha\beta = \sigma_{1,1}$$

We have that $\text{Sym}^2(L_1 \oplus L_2) \cong L_1^{\otimes 2} \oplus (L_1 \otimes L_2) \oplus L_2^{\otimes 2}$, so

$$c(\text{Sym}^2(Q)) = (1 + 2\alpha)(1 + \alpha + \beta)(1 + 2\beta)$$

which gives the degree 3 term $2\alpha\alpha 2\beta + 2\alpha\beta 2\beta = 4(\alpha^2\beta + \alpha\beta^2)$. Substituting in the Schubert cycles, we get

$$c_3(\text{Sym}^2(Q)) = 4(\alpha^2\beta + \alpha\beta^2) = 4(\alpha + \beta)\sigma_{1,1} = 4\sigma_1\sigma_{1,1}$$

By Pieri's Formula above, this gives us $4\sigma_{2,1}$. Therefore we want to find $\int_{G(2,4)} 4\sigma_{2,1}\sigma_1$ which, again by Pieri's Formula, gives $\int_{G(2,4)} 4\sigma_{2,2} = 4$. $\square$

Question 7.10. Calculate the number of lines on a general cubic hypersurface in $\mathbb{P}^3$.

Proof. Following the same approach as in the previous problem, we now seek to find

$$\int_{G(2,4)} c_4(\text{Sym}^3(Q))$$

This is because F this time is a rank-3 bundle, generated by $\{x_i^3, x_i^2x_j, x_ix_j^2, x_j^3\}$ over $U_{x_ix_j}$. $\text{Sym}^3(L_1 \oplus L_2) \cong L_1^{\otimes 3} \oplus (L_1^{\otimes 2} \otimes L_2) \oplus (L_1 \otimes L_2^{\otimes 2}) \oplus L_2^{\otimes 3}$. Calculating this Chern class, we get

$$c(\text{Sym}^3(Q)) = (1 + 3\alpha)(1 + 2\alpha + \beta)(1 + \alpha + 2\beta)(1 + 3\beta)$$

The fourth order term is $18\alpha^3\beta + 45\alpha^2\beta^2 + 18\alpha\beta^3$. Calculating this out in terms of Schubert cycles, we get

$$\begin{aligned} 18\alpha^3\beta + 45\alpha^2\beta^2 + 18\alpha\beta^3 &= 9\alpha\beta(2\alpha^2 + 5\alpha\beta + 2\beta^2) \\ &= 9\alpha\beta(\alpha\beta + 2(\alpha^2 + 2\alpha\beta + \beta^2)) \\ &= 9\alpha\beta(\alpha\beta + 2(\alpha + \beta)^2) \\ &= 9\sigma_{1,1}(\sigma_{1,1} + 2\sigma_1^2) \\ &= 9\sigma_{2,2} + 18\sigma_{2,2} = 27\sigma_{2,2} \end{aligned}$$

Evaluating our integral, we therefore get $\int_{G(2,4)} c_4(\text{Sym}^3(Q)) = 27 \int_{G(2,4)} \sigma_{2,2}$, which gives 27 lines. $\square$

Question 7.11. *Perform the details of the calculation of the number of lines on a quintic threefold.*

Proof. The thing left to prove in the text is evaluating $c_6(Q) = (5\alpha)(4\alpha + \beta)\dots(\alpha + 4\beta)(5\beta)$. I assume the problem is to just bash this out:

$$\begin{aligned}
c_6(Q) &= 5\alpha 4\alpha 3\alpha 2\alpha \alpha 5\beta + 5\alpha \beta 3\alpha 2\alpha \alpha 5\beta \\
&\quad + 5\alpha 4\alpha 3\alpha 2\alpha 4\beta 5\beta + 5\alpha \beta 3\alpha 2\alpha 4\beta 5\beta \\
&\quad + 5\alpha 4\alpha 3\alpha 3\beta \alpha 5\beta + 5\alpha \beta 3\alpha 3\beta \alpha 5\beta \\
&\quad + 5\alpha 4\alpha 3\alpha 3\beta 4\beta 5\beta + 5\alpha \beta 3\alpha 3\beta 4\beta 5\beta \\
&\quad + 5\alpha 4\alpha 2\beta 2\alpha \alpha 5\beta + 5\alpha \beta 2\beta 2\alpha \alpha 5\beta \\
&\quad + 5\alpha 4\alpha 2\beta 2\alpha 4\beta 5\beta + 5\alpha \beta 2\beta 2\alpha 4\beta 5\beta \\
&\quad + 5\alpha 4\alpha 2\beta 3\beta \alpha 5\beta + 5\alpha \beta 2\beta 3\beta \alpha 5\beta \\
&\quad + 5\alpha 4\alpha 2\beta 3\beta 4\beta 5\beta + 5\alpha \beta 2\beta 3\beta 4\beta 5\beta \\
&= 600\alpha^5\beta + (2400 + 900 + 400 + 150)\alpha^4\beta^2 \\
&\quad + (3600 + 1600 + 600 + 600 + 225 + 100)\alpha^3\beta^3 \\
&\quad + (2400 + 900 + 400 + 150)\alpha^2\beta^4 + 600\alpha\beta^5 \\
&= 600\alpha^5\beta + 3850\alpha^4\beta^2 + 6725\alpha^3\beta^3 + 3850\alpha^2\beta^4 + 600\alpha\beta^5 \\
&= 25\alpha\beta(24\alpha^4 + 154\alpha^3\beta + 269\alpha^2\beta^2 + 154\alpha\beta^3 + 24\beta^4) \\
&= 25\alpha\beta(24(\alpha + \beta)^4 + 125\alpha^2\beta^2 + 58(\alpha^3\beta + \alpha\beta^3))
\end{aligned}$$

Substituting in Schubert cycles, we get

$$\begin{aligned}
&= 25\sigma_{1,1}(24\sigma_1^4 + 125\sigma_{1,1}^2 + 58\alpha\beta(\alpha^2 + \beta^2)) \\
&= 25\sigma_{1,1}(24\sigma_1^4 + 125\sigma_{1,1}^2 + 58\sigma_{1,1}((\alpha + \beta)^2 - 2\alpha\beta)) \\
&= 25\sigma_{1,1}(24\sigma_1^4 + 125\sigma_{1,1}^2 + 58\sigma_{1,1}(\sigma_1^2 - 2\sigma_{1,1}))
\end{aligned}$$

Using Pieri's Formula again, we have

$$\begin{aligned}
\sigma_1 \sigma_{1,1} &= \sigma_{2,1} \\
\sigma_1^2 &= \sigma_{1,1} + \sigma_2 \\
\sigma_2 \sigma_{1,1} &= \sigma_{3,1} \\
\sigma_2 \sigma_1 &= \sigma_3 + \sigma_{2,1} \\
\sigma_1 \sigma_{2,1} &= \sigma_{3,1} + \sigma_{2,2} \\
\sigma_1 \sigma_3 &= \sigma_{3,1} \\
\sigma_2^2 &= \sigma_{3,1} + \sigma_{2,2}
\end{aligned}$$

Therefore, we get $\sigma_1^4 = 3\sigma_{3,1} + 2\sigma_{2,2}$ and $\sigma_{1,1}(\sigma_1^2 - 2\sigma_{1,1}) = \sigma_{3,1} + \sigma_{2,2} - 2\sigma_{1,1}^2$. This gives

$$\begin{aligned}
c_6(Q) &= 25\sigma_{1,1}(24\sigma_1^4 + 125\sigma_{1,1}^2 + 58\sigma_{1,1}(\sigma_1^2 - 2\sigma_{1,1})) \\
&= 25\sigma_{1,1}(130\sigma_{3,1} + 48\sigma_{2,2} + 67\sigma_{1,1}^2)
\end{aligned}$$

We can find $\sigma_{1,1}^2$ by using what we know about σ_1^2 and σ_1^4 :

$$\begin{aligned}
(\sigma_1^2)^2 &= \sigma_1^4 \\
(\sigma_{1,1} + \sigma_2)^2 &= 3\sigma_{3,1} + 2\sigma_{2,2} \\
\sigma_{1,1}^2 + 2\sigma_{1,1}\sigma_2 + \sigma_2^2 &= 3\sigma_{3,1} + 2\sigma_{2,2} \\
\Rightarrow \sigma_{1,1}^2 &= \sigma_{2,2}
\end{aligned}$$

So we now have our Chern class as

$$c_6(Q) = 25\sigma_{1,1}(130\sigma_{3,1} + 115\sigma_{2,2})$$

It remains to find $\sigma_{1,1}\sigma_{3,1}$ and $\sigma_{1,1}\sigma_{2,2}$. Starting with the former, knowing that $\sigma_2\sigma_{1,1} = \sigma_{3,1}$, we can multiply both sides by $\sigma_{1,1}$ further to get $\sigma_{3,1}\sigma_{1,1} = \sigma_2\sigma_{2,2}$. Using Pieri's Formula once again, we see that this is equal to 0. For the latter, we see that

$$\begin{aligned}
\sigma_1^2 &= \sigma_{1,1} + \sigma_2 \\
\Rightarrow \sigma_1^2 \sigma_{2,2} &= \sigma_{1,1}\sigma_{2,2} + \sigma_2\sigma_{2,2}
\end{aligned}$$

Using Pieri's Formula yet again, we have that $\sigma_1^2 \sigma_{2,2} = \sigma_{3,3} = \sigma_{1,1}\sigma_{2,2}$. Thus we have $c_6(Q) = 2875\sigma_{3,3}$. Integrating this over $G(2, 5)$, we get 2875 lines as desired. $\square$

8 Excess Intersection

Question 8.1. Assuming the validity of the Splitting Principle, prove the formula

$$c(E/F) = \frac{c(E)}{c(F)}$$

for the Chern classes of a quotient bundle.

Proof. Recall that the Splitting Principle states that a rank- r vector bundle can be thought of as being able to be decomposed into a direct sum of r line bundles.

We know from the text that $c(L_1 \oplus \dots \oplus L_r) = \prod_{i=1}^r (1 + c_1(L_i))$. Choose a basis of E which we'll label as $\{x_1, \dots, x_n\}$, where x_i is the basis of L_i . Our basis of a subbundle F will be given by $\{x_1, \dots, x_k\}$, $k \leq n$, so the basis of E/F is given by $\{x_{k+1}, \dots, x_n\}$. Then we have

$$\frac{c(E)}{c(F)} = \frac{\prod_{i=1}^r (1 + c_1(L_i))}{\prod_{i=1}^k (1 + c_1(L_i))} = \prod_{i=k+1}^r (1 + c_1(L_i))$$

This is the same Chern class of the $(L_{k+1} \oplus \dots \oplus L_n)$ bundle, which by the Splitting principle we set as equivalent to E/F . $\square$

Question 8.2. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d , and let N be the normal bundle of X in $\mathbb{P}^n$. Prove that $N \simeq \mathcal{O}_{\mathbb{P}^n}(d)|_X$. Hint: let F be a homogeneous polynomial of degree d with $X = Z(F)$, and consider the map

$$\psi : T\mathbb{P}^n|_X \rightarrow \mathcal{O}_{\mathbb{P}^n}(d)|_X, \quad \psi(\theta) = \theta(F)|_X.$$

Show that $\psi(TX) = 0$ and that ψ induces an isomorphism from N to $\mathcal{O}_{\mathbb{P}^n}(d)|_X$.

Proof. First we remind ourselves that, proven in the final exercise of Chapter 6, $\mathcal{O}_{\mathbb{P}^n}(d)$ has sections of inhomogeneous degree d polynomials: If U_α is our standard chart on $\mathbb{P}^n$ ($c_\alpha \neq 0$), i.e. coordinates on $U_\alpha \subset \mathbb{P}^n$ are given by

$$\left(\frac{c_0}{c_\alpha}, \dots, \frac{c_{\alpha-1}}{c_\alpha}, \frac{c_{\alpha+1}}{c_\alpha}, \dots, \frac{c_n}{c_\alpha}\right) := (z_1, \dots, z_n) \in U_\alpha \subset \mathbb{P}^n$$

with sections $\mathbb{P}^n \rightarrow \mathcal{O}_{\mathbb{P}^n}(d)$ and transition functions given by

$$G(z_1, \dots, z_n), \quad g_{\beta\alpha} = \left(\frac{c_\alpha}{c_\beta}\right)^d$$

where $G(z_1, \dots, z_n)$ is a homogeneous polynomial of degree d .

We follow the hint and let F be the homogeneous polynomial with $X = Z(F)$.

First we prove well-definedness. In $U_\alpha \cap X$ we'll label our coordinates $(z_1, \dots, z_n)$ and in $U_\beta \cap X$ we'll label our coordinates $(z'_1, \dots, z'_n)$. For $U_\alpha \cap U_\beta \cap X$ we have

$$\begin{aligned}\theta_\alpha &:= \sum_{i=1}^n f_i \frac{\partial}{\partial z_i} \\ &= \sum_{i,j=1}^n f_i \frac{\partial z'_j}{\partial z_i} \frac{\partial}{\partial z'_j} \\ &= g_{\alpha\beta} \theta_\beta\end{aligned}$$

From the Chain Rule,

$$\frac{\partial F(z_1, \dots, z_n)}{\partial z_i} = \frac{\partial z'_j}{\partial z_i} \frac{\partial F(z'_1(z_1, \dots, z_n), \dots, z'_n(z_1, \dots, z_n))}{\partial z'_j},$$

so $\psi(\theta_\alpha) = \psi(g_{\alpha\beta} \theta_\beta)$ on $U_\alpha \cap U_\beta \cap X$. Therefore, ψ is well-defined.

For each section $\theta = \sum_i f_i \frac{\partial}{\partial z_i}$ of $T\mathbb{P}^n|_X$, ψ gives us the vector field $\sum_i f_i \frac{\partial F}{\partial z_i}$.

Note that $(\frac{\partial F}{\partial z_i})$ is the gradient of F , which is normal to X . Therefore, if (f_i) is a vector tangent to X , i.e. $\in TX$, $(f_i) \cdot (\frac{\partial F}{\partial z_i}) = 0$. Thus $\psi|_{TX} = 0$. Alternatively, we note that, since ψ is coordinate-independent and F gives a section s of $\mathcal{O}_{\mathbb{P}^n}(d)$, s can be locally expressed as

$$s = (0, \dots, 0, z_n),$$

with $Z(F) = X$ having coordinates $(z_1, \dots, z_{n-1})$. Elements of TX are given by $(f_1 \frac{\partial}{\partial z_1}, \dots, f_{n-1} \frac{\partial}{\partial z_{n-1}}, 0)$, so we have

$$\theta(F)|_X = \sum_{i=1}^{n-1} \frac{\partial s}{\partial z_i} = \sum_{i=1}^{n-1} \frac{\partial}{\partial z_i} (0, \dots, 0, z_n) = 0$$

Thus $\psi(TX) = 0$.

Next we prove isomorphism. Denote F expressed in U_α coordinates by F_α . F_α can be expressed as $\frac{1}{x_\alpha^d} f_\alpha(x_1, \dots, x_{\alpha-1}, x_{\alpha+1}, \dots, x_n)$, where f_α is a homogeneous

degree d polynomial in $n - 1$ variables. Therefore, transition functions $g'_{\alpha\beta}$ from F_α to F_β are given by $(\frac{x_\alpha}{x_\beta})^d$. We then have

$$\begin{aligned}\psi(\theta_\beta) &= \theta_\beta(F_\beta)|_X \\ &= g_{\alpha\beta}\theta_\alpha(g'_{\alpha\beta}F_\alpha)|_X \\ &= g_{\alpha\beta}[\theta_\alpha(g'_{\alpha\beta})F_\alpha|_X + g'_{\alpha\beta}\theta_\alpha(F_\alpha)|_X] \\ &= g_{\alpha\beta}[0 + g'_{\alpha\beta}\theta_\alpha(F_\alpha)|_X] \\ &= (\frac{x_\alpha}{x_\beta})^d g_{\alpha\beta}\theta_\alpha(F_\alpha)|_X\end{aligned}$$

The $(\frac{x_\alpha}{x_\beta})^d$ are exactly the transition functions of $\mathcal{O}_{\mathbb{P}^n}(d)$, so ψ induces an isomorphism from $T\mathbb{P}^n/TX|_X = N$ to $\mathcal{O}_{\mathbb{P}^n}(d)|_X$. $\square$

Question 8.3. Check that the map $\mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \rightarrow T\mathbb{P}^n$ defined by

$$(l_0, \dots, l_n) \mapsto \sum_{i=0}^n l_i \frac{\partial}{\partial x_i}$$

is onto and identifies $T\mathbb{P}^n$ with the quotient of $\mathcal{O}_{\mathbb{P}^n}(1)^{n+1}$ by a trivial line subbundle.

Proof. We will choose local coordinates in U_α , which we'll label $x = (x_0, \dots, x_n)$. Suppose we have an arbitrary element $\sum_{i=0}^n a_i \frac{\partial}{\partial x_i}$ in $T_x\mathbb{P}^n$. Since we have $n + 1$ coefficients a_i , we can find n^2 values l_i such that $l_i(x) = a_i$ for all i :

$$\begin{pmatrix} l_0(x) \\ \vdots \\ l_n(x) \end{pmatrix} := \begin{pmatrix} l_{00} & \dots & l_{0n} \\ \vdots & \ddots & \vdots \\ l_{n0} & \dots & l_{nn} \end{pmatrix} \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix}$$

Thus we can find an element $(l_0, \dots, l_n) \in \mathcal{O}_{\mathbb{P}^n}(1)^{n+1}$ such that the image of this map is our arbitrarily selected element of $T\mathbb{P}^n$. Therefore the mapping is onto.

We saw in the text that the trivial line subbundle F generated by $(x_0, \dots, x_n) \in \mathcal{O}_{\mathbb{P}^n}(1)^{n+1}$ maps to 0 in $T\mathbb{P}^n$ as a consequence of Euler's Formula, i.e. is the kernel of this mapping. Then by the First Isomorphism Theorem of modules, since $T\mathbb{P}^n$ is the image of this mapping, $T\mathbb{P}^n \cong \mathcal{O}_{\mathbb{P}^n}(1)^{n+1}/F$. $\square$

Question 8.4. Let Q_1, Q_2, Q_3 be degree 2 hypersurfaces in $\mathbb{P}^3$ which contain a line L but are otherwise general. Suppose that $Q_1 \cap Q_2 \cap Q_3 = L \cup \{p_1, \dots, p_k\}$. Find k .

Proof. Since Q_1, Q_2 , and Q_3 are general, we assume they aren't singular or double lines. We view L as $\mathbb{P}^1$ in $\mathbb{P}^3$. We denote $H \in H^2(\mathbb{P}^3), h \in H^2(\mathbb{P}^1)$ as the hyperplane classes of $\mathbb{P}^3, \mathbb{P}^1$, respectively. Note that the restriction of $\mathcal{O}_{\mathbb{P}^3}(1)|_{\mathbb{P}^1}$ is $\mathcal{O}_{\mathbb{P}^1}(1)$, so the restriction of H to $\mathbb{P}^1$ is h . Calculating the Chern class of the normal bundle $N_{L/\mathbb{P}^3}$ yields

$$\frac{c(T\mathbb{P}^3|_{\mathbb{P}^1})}{c(T\mathbb{P}^1)} = \frac{(1+H)^4|_{\mathbb{P}^1}}{(1+h)^2} = \frac{(1+h)^4}{(1+h)^2} = (1+h)^2 = 1+2h+h^2 = 1+2h$$

We view (Q_1, Q_2, Q_3) as a section of $\mathcal{O}_{\mathbb{P}^3}(2)^3$, and then compute the Chern class of the excess bundle $B = \mathcal{O}_{\mathbb{P}^3}(2)^3|_{\mathbb{P}^3}/N_{L/\mathbb{P}^3}$:

$$c(B) = \frac{c(\mathcal{O}_{\mathbb{P}^3}(2)^3|_{\mathbb{P}^1})}{c(N_{L/\mathbb{P}^3})} = \frac{(1+2H)^3|_{\mathbb{P}^1}}{1+2h} = \frac{(1+2h)^3}{1+2h} = 1+4h$$

Therefore $c_1(B) = 4h$ and $\int_{\mathbb{P}^1} c_1(B) = 4$.

Bezout's Theorem gives $2*2*2 = 8$ points of intersection. Since we calculated that 4 of these are due to our conics' common line, there must be 4 extra points of intersection. $\square$

9 Rational Curves on the Quintic Threefold

Question 9.1. Find all automorphisms of the stable map $f : C \rightarrow \mathbb{P}^4$ described as follows. The curve C has two components C_1 and C_2 , with restrictions $4f_i$ of f to C_i given by

$$\begin{aligned} f_1(x_0, x_1) &= (x_0^3, x_1^3, 0, 0, 0) \\ f_2(x_0, x_1) &= (x_1^3, x_0^3, 0, 0, 0). \end{aligned}$$

We have described f_i by identifying each C_i with $\mathbb{P}^1$. The curve C has a single node, formed by joining the point $(0,1)$ of C_1 to the point $(1,0)$ of C_2 .

Proof. We wish to find automorphisms $g(x_0, x_1) \mapsto (a_0x_0 + a_1x_1, b_0x_0 + b_1x_1)$ such that their images, when cubed, give x_0^2 and x_1^3 or vice versa. Expanding this out, this means we require either $a_0^3 = b_1^3 = 1, a_1 = b_0 = 0$, or $a_1^3 = b_0^3 = 1, a_0 = b_1 = 0$. Recall that we're working in $\mathbb{C}$, so these coefficients are allowed to take third roots of unity $e^{i2\pi/3}$ and their powers. Assigning each of $a_0, b_1 \in \{e^{i2\pi/3}, e^{i4\pi/3}, 1\}$ in the case where $a_1 = b_0 = 0$ yields an automorphism group $\mathbb{Z}_3 \times \mathbb{Z}_3$. Including the other case adds a factor of $\mathbb{Z}_2$, as it flips the values $a_0 \leftrightarrow a_1, b_0 \leftrightarrow b_1$, so the automorphism group is $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$. $\square$

Question 9.2. It was asserted that E_5 is a rank 11 vector bundle on $\overline{M}(\mathbb{P}^4, 2)$. Check that $\dim(E_5)_f = 11$ for the stable map $f : C \rightarrow \mathbb{P}^4$ described as follows. The curve C has two components C_1 and C_2 , with restrictions f_i of f to C_i given by

$$f_1(x_0, x_1) = (x_0, x_1, 0, 0, 0)$$

and

$$f_2(x_0, x_1) = (0, x_0, x_1, 0, 0).$$

We have described f_i by identifying each C_i with $\mathbb{P}^1$. The curve C has a single node, formed by joining the point $(0, 1)$ of C_1 to the point $(1, 0)$ of C_2 .

Proof. In the text, $(E_5)_f$ is defined as

$$(E_5)_f = H^0(f^*(\mathcal{O}_{\mathbb{P}^4}(5)))$$

which is the vector space of sections of the pullback of $\mathcal{O}_{\mathbb{P}^4}(5)$ to C .

Sections of $\mathcal{O}_{\mathbb{P}^4}(5)$ correspond to degree 5 homogeneous polynomials over $\mathbb{P}^4$. We choose an arbitrary section $\sum_{|I|=5} c_I x^I$. There are two variables in $\mathbb{P}^1$ among

which we need to choose five terms with replacement where order doesn't matter, i.e. $|c_I| = \binom{n+d}{d} = \binom{9}{5} = 3024$ terms in this polynomial. The pullback corresponds to setting n to 1, as the preimage of each of the components of f is $\mathbb{P}^1$, so there are $\binom{1+d}{d} = d + 1$ coefficients. Since there are two components, we have a $2(d + 1)$ coefficients. However, one of them is shared, as there is one node, so the vector space of sections is $2(d + 1) - 1 = 2d + 1$ -dimensional.

We can see this explicitly for f : The pullback via f corresponds to setting $x_3 = x_4 = 0$ and either $x_2 = 0$ or $x_0 = 0$ in the polynomial, so that only the terms with coefficients either $c_{..000}$ or $c_{0..00}$ are kept:

$$\begin{aligned} & c_{50000}x_0^5 + c_{41000}x_0^4x_1 + c_{32000}x_0^3x_1^2 + c_{23000}x_0^2x_1^3 + c_{14000}x_0^1x_1^4 + c_{05000}x_1^5, \\ & c_{05000}x_0^5 + c_{04100}x_0^4x_1 + c_{03200}x_0^3x_1^2 + c_{02300}x_0^2x_1^3 + c_{01400}x_0^1x_1^4 + c_{00500}x_1^5 \end{aligned}$$

Thus we have 11 unique (c_{05000} shows up twice) coefficients that are kept in the pullback. These correspond to sections of the pullback $f^*(\mathcal{O}_{\mathbb{P}^4}(5))$, so this is an 11-dimensional vector space. $\square$

Question 9.3. Now prove that for any $f \in \overline{M}(\mathbb{P}^4, d)$, we have $\dim(E_5)|_f = 5d + 1$.

Proof. From the previous problem, we have that sections of each component f_i correspond to $\mathcal{O}_{\mathbb{P}^1}(5c_i)$, where c_i is the degree of f_i : if $f_i : \mathbb{P}^1 \rightarrow \mathbb{P}^4$ defines an irreducible rational curve of degree c_i , then $f_i^*(\mathcal{O}_{\mathbb{P}^4}(c_i)) \simeq \mathcal{O}_{\mathbb{P}^1}(5c_i)$ since degree c_i polynomials in the variables of $\mathbb{P}^4$ pull back via f_i to polynomials of degree $5c_i$ in the variables of $\mathbb{P}^1$. Each of these sections are classified by $\binom{1+5c_i}{5c_i} = 5c_i + 1$ coefficients corresponding to the terms in the homogeneous polynomials. Suppose our stable map has m components. For each component, we then have $5c_i + 1$ coefficients, so we have $\sum_{i=1}^m (5c_i + 1)$ coefficients in total. However, we must subtract the number of nodes, as these are shared coefficients. For m components, we have $m - 1$ nodes which we must subtract, so we have

$$\begin{aligned} \sum_{i=1}^m (5c_i + 1) - (m - 1) &= 5 \sum_{i=1}^m c_i + m - m + 1 \\ &= 5d + 1 \end{aligned}$$

unique sets of coefficients, and therefore a $(5d + 1)$ -dimensional vector space. $\square$

10 Mechanics

In the first two exercises, 1-dimensional Newtonian mechanics is generalized to n dimensions. A path is given in coordinates by a vector-valued function $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))$. Introduce a potential function $V(\mathbf{x})$. The action is

$$S = \int_{t_0}^{t_1} \left(\frac{m}{2} \sum_{i=1}^n \left(\frac{dx_i}{dt} \right)^2 - V(\mathbf{x}) \right) dt$$

Question 10.1. Derive the equations of motion $md^2x_i/dt^2 + \partial V/\partial x_i = 0$. If we put $\mathbf{F} = -\nabla V$, this is just the familiar $\mathbf{F} = m\mathbf{a}$.

Proof. The Lagrangian $\mathcal{L}$ is the integrand, and the Euler-Lagrange Equations give

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta x_i(t)} &= \frac{d}{dt} \left(\frac{\delta \mathcal{L}}{\delta x'_i(t)} \right) \\ \frac{\partial V}{\partial x_i} &= \frac{d}{dt} \left(m \frac{dx_i}{dt} \right) \\ \frac{\partial V}{\partial x_i} &= m \frac{d^2 x_i}{dt^2} \end{aligned}$$

for each $x_i(t)$. Summing both sides over all $x_i(t)$, we get $\nabla V + m\mathbf{a} = 0$. $\square$

Question 10.2. For $n = 3$, let $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and suppose that $V = V(r)$. Show that the action is invariant under rotations $\mathbf{x} \rightarrow \mathbf{x}' = A\mathbf{x}$, where A is the 3×3 matrix of a rotation (which satisfies $AA^t = I$, the identity matrix) and $\mathbf{x}, \mathbf{x}'$ are expressed as column vectors. What quantity is conserved by rotation about the third axis, i.e., by the rotations with matrix

$$A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}^*.$$

*The center entry in my copy of the book says $\sin(\theta)$ but it should be $\cos(\theta)$.

Proof. The only term in the Lagrangian concerning $x_i(t)$ is the potential function. Note that we have $V(r) = V(\sqrt{\mathbf{x}^t \mathbf{x}})$. Therefore, with the transformation $\mathbf{x} \rightarrow A\mathbf{x}$, we have $V(r) \rightarrow V(\sqrt{(A\mathbf{x})^t A\mathbf{x}}) = V(\sqrt{\mathbf{x}^t A^t A \mathbf{x}}) = V(\sqrt{\mathbf{x}^t \mathbf{x}}) = V(r)$. Taking an infinitesimal rotation, we Taylor Expand our rotation to first order:

$$\begin{aligned} x_1 &\rightarrow x_1 - \theta x_2 \\ x_2 &\rightarrow \theta x_1 + x_2 \\ x_3 &\rightarrow x_3 \end{aligned}$$

Thus our potential function gets transformed to first order as

$$V(\mathbf{x}) \rightarrow V(A\mathbf{x}) \tag{2}$$

$$\rightarrow V(\mathbf{x}) + \sum_{i=1}^3 \frac{\partial V}{\partial x_i} (Ax_i - x_i) \tag{3}$$

$$= V(\mathbf{x}) + \frac{\partial V}{\partial x_1} (-\theta x_2) + \frac{\partial V}{\partial x_2} (\theta x_1) \tag{4}$$

$$= V(\mathbf{x}) - \theta (\nabla V \times \mathbf{x})_{2D} \tag{5}$$

$$= V(\mathbf{x}) - \theta (m\mathbf{a} \times \mathbf{x})_{2D} \tag{6}$$

Since our potential is the only term in our action that depends on $\mathbf{x}$, the variation in our action is

$$\begin{aligned} \delta S &= - \int (\theta m\mathbf{a} \times \mathbf{x})_{2D} dt \\ &= -\theta m\mathbf{v} \times \mathbf{x} \Big|_{t_0}^{t_1} \\ &= -\theta \mathbf{p} \times \mathbf{x} \Big|_{t_0}^{t_1} \end{aligned}$$

Since $\delta S = 0$, the $\mathbf{p} \times \mathbf{x}$ (in dimensions 1 and 2, in this case) must be the same at t_0 and t_1 , so the $\mathbf{p} \times \mathbf{x}$ is conserved. This is canonically angular momentum. $\square$

Question 10.3. *a Check that the infinitesimal transformation*

$$\delta x(t) = -\epsilon \frac{dx}{dt}$$

is a symmetry of the action given by

$$S = \int_{t_0}^{t_1} \left(\frac{m}{2} \left(\frac{dx}{dt} \right)^2 - V(x(t)) \right) dt$$

Use Noether's method applied to the above infinitesimal transformation to derive the conservation of total energy ($H(t) = \frac{1}{2}m \left(\frac{dx}{dt} \right)^2 + V(x)$) of a 1-dimensional particle in the presence of a potential $V(x)$. In the above, ϵ is an infinitesimal parameter.

b Check directly that the total energy is conserved, using Newton's Law ($m \frac{d^2x}{dt^2} + V'(x) = 0$).

Proof. *a Notice that $x - \epsilon \frac{dx}{dt}$ is the Taylor expansion of $x(t + \epsilon)$. Since t is unchanged, the position function doesn't change, and the Lagrangian just experiences a change of variables - shifting t in the function doesn't change the integral. Therefore the infinitesimal transformation is a symmetry.*

Substituting $x + \delta x$ into our action, we get

$$\begin{aligned} S' &= \int_{t_0}^{t_1} \left(\frac{m}{2} \left(\frac{dx}{dt} - \epsilon \frac{d^2x}{dt^2} \right)^2 - V\left(x - \epsilon \frac{dx}{dt}\right) \right) dt \\ &\approx \int_{t_0}^{t_1} \left(\frac{m}{2} \left(\left(\frac{dx}{dt} \right)^2 - 2\epsilon \frac{d^2x}{dt^2} \frac{dx}{dt} \right)^2 - V\left(x - \epsilon \frac{dx}{dt}\right) \right) dt \end{aligned}$$

where we remove higher order terms of ϵ . Taylor expanding the potential term, we get

$$\begin{aligned} S' &= \int_{t_0}^{t_1} \left(\frac{m}{2} \left(\left(\frac{dx}{dt} \right)^2 - 2\epsilon \frac{d^2x}{dt^2} \frac{dx}{dt} \right) - V(x) + \epsilon \frac{\partial V}{\partial x} \frac{dx}{dt} \right) dt \\ &= \int_{t_0}^{t_1} \left(\frac{m}{2} \left(\left(\frac{dx}{dt} \right)^2 - 2\epsilon \frac{d^2x}{dt^2} \frac{dx}{dt} \right) - V(x) + \epsilon \frac{\partial V}{\partial x} \right) dt \end{aligned}$$

Subtracting our action that had the original x position, we get

$$\begin{aligned}\delta S &= \int_{t_0}^{t_1} \left(-\epsilon \frac{m}{2} 2 \frac{d^2 x}{dt^2} \frac{dx}{dt} + \epsilon \frac{\partial V}{\partial t} \right) dt \\ &= \int_{t_0}^{t_1} \epsilon \left(\frac{\partial}{\partial t} \left(\frac{m}{2} \frac{dx}{dt} \frac{dx}{dt} \right) + \frac{\partial V}{\partial t} \right) dt \\ &= \epsilon \left[\frac{m}{2} \left(\frac{dx}{dt} \right)^2 + V(x) \right]_{t_0}^{t_1}\end{aligned}$$

Since $\delta S = 0$, this means that the quantity $\frac{m}{2} \left(\frac{dx}{dt} \right)^2 + V(x)$ is constant.

- b Taking the time derivative of the total energy $H(t) = \frac{1}{2}m \left(\frac{dx}{dt} \right)^2 + V(x)$, we get

$$\begin{aligned}\frac{\partial H(t)}{\partial t} &= m \frac{dx}{dt} \frac{d^2 x}{dt^2} + \frac{\partial V(x)}{\partial t} \\ &= -\frac{\partial V(x)}{\partial x} \frac{dx}{dt} + \frac{\partial V(x)}{\partial t}\end{aligned}$$

by Newton's Law. By the Chain rule we have

$$-\frac{\partial V(x)}{\partial x} \frac{dx}{dt} + \frac{\partial V(x)}{\partial t} = -\frac{\partial V(x)}{\partial t} + \frac{\partial V(x)}{\partial t} = 0$$

Therefore total energy is constant i.e. conserved.

□

Question 10.4. Give a very crude approximation for the path integral using

$$S = \int_{t_0}^{t_1} \left(\frac{m}{2} \left(\frac{dx}{dt} \right)^2 - V(x(t)) \right) dt$$

as follows. Pick an integer N , rename the right-hand endpoint of the interval to be t_N , put $\Delta t = (t_N - t_0)/N$, and divide $[t_0, t_N]$ into N intervals $[t_i, t_{i+1}]$ of length Δt . Put $x_i = x(t_i)$ for $1 \leq i \leq N-1$. Approximate the path integral as an appropriate finite-dimensional integral P_N over the ordinary variables $x_1, \dots, x_{N-1}$.

The path integral can be “defined” by letting $N \rightarrow \infty$.

Proof. We begin by splitting up the action into these time intervals. Define

$$\begin{aligned} S_i &:= \int_{t_{i-1}}^{t_i} \left(\frac{m}{2} \left(\frac{x_i - x_{i-1}}{\Delta t} \right)^2 - V(x_i - x_{i-1}) \right) dt \\ &\approx \frac{m}{2\Delta t} (x_i - x_{i-1})^2 - V(x_i - x_{i-1})\Delta t \end{aligned}$$

The full action S is the sum of the S_i , so our action under this approximation is

$$S = \sum_{i=1}^{N-1} \frac{m}{2\Delta t} (x_i - x_{i-1})^2 - V(x_i - x_{i-1})\Delta t$$

Each of the x_i , $1 \leq i \leq N-1$ can range over the entire 1-dimensional domain, $(-\infty, \infty)$, so our path integral approximation P_N becomes

$$P_N = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left(\sum_{i=1}^{N-1} \frac{m}{2\Delta t} (x_i - x_{i-1})^2 - V(x_i - x_{i-1})\Delta t \right) dx_1 \dots dx_{N-1}$$

□

11 Introduction to Supersymmetry

Question 11.1. If the ψ_i are anticommuting variables, express $e^{x^2 + x\psi_1\psi_2 - \psi_3\psi_4}$ as a polynomial in the ψ_i whose coefficients are functions of x .

Proof. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we factor our exponential into

$$\begin{aligned} e^{x^2 + x\psi_1\psi_2 - \psi_3\psi_4} &= e^{x^2} e^{x\psi_1\psi_2} e^{-\psi_3\psi_4} \\ &= e^{x^2} \left(\sum_{n=0}^{\infty} \frac{(x\psi_1\psi_2)^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{(-\psi_3\psi_4)^n}{n!} \right) \\ &= e^{x^2} (1 + x\psi_1\psi_2) (1 - \psi_3\psi_4) \end{aligned}$$

since $(\psi_i\psi_j)^n = 0$ for $n \geq 2$. Therefore our polynomial in ψ_i is

$$e^{x^2} - e^{x^2} \psi_3\psi_4 + xe^{x^2} \psi_1\psi_2 - xe^{x^2} \psi_1\psi_2\psi_3\psi_4$$

□

Question 11.2. Consider a change of fermionic coordinates

$$\begin{aligned}\psi'_1 &= a\psi_1 + b\psi_2 \\ \psi'_2 &= c\psi_1 + d\psi_2\end{aligned}$$

Prove the change of variables formula

$$\int f(\psi'_1, \psi'_2) d\psi'_1 d\psi'_2 = \int f(a\psi_1 + b\psi_2, c\psi_1 + d\psi_2) (ad - bc)^{-1} d\psi_1 d\psi_2.$$

Proof. Subbing in $\psi'_1 = a\psi_1 + b\psi_2, \psi'_2 = c\psi_1 + d\psi_2$, we get the substitution in the f term in the integral, and we also get

$$\begin{aligned}d\psi'_1 &= ad\psi_1 + bd\psi_2 \\ d\psi'_2 &= cd\psi_1 + dd\psi_2\end{aligned}$$

Therefore $d\psi'_1 d\psi'_2 = (ad\psi_1 + bd\psi_2)(cd\psi_1 + dd\psi_2)$. Multiplying this out, we get $acd\psi_1^2 + add\psi_1 d\psi_2 + bcd\psi_2 d\psi_1 + bdd\psi_2^2$. Due to anticommutativity, $d\psi_1^2 = d\psi_2^2 = 0$, so $d\psi'_1 d\psi'_2$ is equal to $add\psi_1 d\psi_2 + bcd\psi_2 d\psi_1 = (ad - bc)d\psi_1 d\psi_2$. Thus $d\psi'_1 d\psi'_2 = \frac{1}{ad-bc} d\psi_1 d\psi_2$ by Berezin integration given in the book. $\square$

Question 11.3. Let A be a $2n \times 2n$ skew-symmetric matrix, and let

$$\psi = (\psi_1, \dots, \psi_{2n})$$

be a vector of mutually anticommuting variables. Prove that

$$\int \frac{1}{2} e^{-\psi A(\psi)^t} d\psi_1 \dots d\psi_{2n} = Pf(A),$$

where $Pf(A)$ is the Pfaffian of A , which satisfies $Pf(A)^2 = \det(A)$.

Proof. Since A is skew-symmetric, $\psi A \psi^t = \sum_{1 \leq i < j \leq 2n} 2a_{ij} \psi_i \psi_j$ due to anticommutativity. Therefore $\frac{1}{2} e^{-\psi A \psi^t} = \frac{1}{2} \prod_{1 \leq i < j \leq 2n} (1 - 2a_{ij} \psi_i \psi_j)$ since $(\psi_i \psi_j)^{\geq 2} = 0$. This product will result in many multiplications of our fermionic variables, but the only terms that will stay after fermionic integration over *all* ψ_i will be the terms with all $2n$ ψ_i s, as other terms will be constant with respect to fermionic integration over a ψ_j that isn't contained in the term. The terms that stay will thus be a signed sum

of over all permutations $i_1, \dots, i_{2n}$ for $2^n a_{i_1 i_2} \dots a_{i_{2n-1} i_{2n}} \psi_{i_1} \dots \psi_{i_{2n}}$. Integrating these terms gives the sum

$$\begin{aligned} 2^n \sum_{(i_1, \dots, i_{2n}) \in \text{Sym}_{2n}} a_{i_1 i_2} \dots a_{i_{2n-1} i_{2n}} \int \epsilon_{i_1, \dots, i_{2n}} d\psi_1 \dots d\psi_{2n} \\ = 2^n \sum_{(i_1, \dots, i_{2n}) \in \text{Sym}_{2n}} \epsilon_{i_1, \dots, i_{2n}} a_{i_1 i_2} \dots a_{i_{2n-1} i_{2n}} \end{aligned}$$

where ϵ is the Levi-Civita symbol for $2n$ elements. This is exactly the Pfaffian. $\square$

Question 11.4. Check that the direct calculation

$$\begin{aligned} Z &= \int \exp(S(x, \psi_1, \psi_2)) dx d\psi_1 d\psi_2 \\ &= \int \exp(-h'(x)^2/2 + h''(x)\psi_1\psi_2) dx d\psi_1 d\psi_2 \\ &= c\sqrt{\pi}, \quad c \in \{0, 1, -1\} \end{aligned}$$

of the $(0+1)$ -dimensional supersymmetric path integral agrees with the localization calculation

$$\frac{Z}{\sqrt{\pi}} = \sum_{h'(x_i)=0} \epsilon_i, \quad \epsilon_i = \begin{cases} +1 & h''(x_i) > 0 \\ -1 & h''(x_i) < 0. \end{cases}$$

in all cases.

Proof. In the direct calculation, if h' has even degree, then $c = 0$ as $h'(-\infty) = h'(\infty)$. If h' has odd degree, then $c = \pm 1$, the sign being the same as that of the leading coefficient a of h' , as $h'(\pm\infty) = (\pm \text{sign}(a))\infty$.

Note that, if there are two adjacent zeros of h' , then h'' must change sign between these two points - if not, then h'' wouldn't be able to cross the zero line again after the first zero.

Suppose that h' has even degree. Then there are an even number of solutions x_i to $h'(x_i) = 0$. As outlined in the above note, every zero of h' is followed by another zero where h'' is of opposite sign. An even number of zeros means that the $+1$ ϵ_i s are equal in number of the -1 ϵ_i s, giving $\frac{Z}{\sqrt{\pi}} = 0$ if h' has even degree.

Suppose that h' has odd degree with leading coefficient $a > 0$. Then h' has an odd number of zeros and $h' \rightarrow -\infty$ as $x \rightarrow -\infty$. Thus h'' is increasing until

the most negative zero of h' . Therefore the leftmost zero contributes $\epsilon_i = +1$ to the sum. Since there are an odd number of zeros and the corresponding ϵ_i s are alternating, this sum ends with a +1.

The case where h' has odd degree with leading coefficient $a < 0$. Then h'' is decreasing until the most negative zero of h' . By similar reasoning, the sum ends with a -1. $\square$

Question 11.5. *Prove the change of variables formula needed to justify the equality*

$$Z' = \int \exp(-S(\tilde{x}, 0, 0)) d\tilde{x} d\tilde{\psi}_1 d\tilde{\psi}_2$$

Show how this leads to the result that Z is a sum over the critical points of the local contributions

$$\tilde{Z} = \pm \sqrt{\pi}$$

Proof. The change of variables is

$$(\tilde{x}, \tilde{\psi}_1, \tilde{\psi}_2) = (x - \frac{\psi_1 \psi_2}{h'(x)}, \psi_1, \psi_2)$$

The action given in our original coordinates is the following, where we Taylor expand then square:

$$\begin{aligned} S(x, \psi_1, \psi_2) &= \frac{h'(x - (\psi_1 \psi_2 / h'(x)))^2}{2} \\ &= \frac{1}{2} \left(h'(x) - h''(x) \frac{\psi_1 \psi_2}{h'(x)} \right)^2 \\ &= \frac{h'(x)^2}{2} - h''(x) \psi_1 \psi_2 \\ e^{-S} &= e^{-\frac{h'(x)^2}{2} + h''(x) \psi_1 \psi_2} \\ &= e^{-\frac{h'(x)^2}{2}} (1 + h''(x) \psi_1 \psi_2) \end{aligned}$$

The book claims that this change of variables satisfies $d\tilde{x}d\tilde{\psi}_1d\tilde{\psi}_2 = dx d\psi_1 d\psi_2$, and even explicitly states that we need the omitted definition of the Berezin integral to justify this. So I'm not sure what he means, but I'll just say that

$$\int e^{-S(x, \psi_1, \psi_2)} dx d\psi_1 d\psi_2 = \int e^{-\frac{h'(x)^2}{2}} d\tilde{x} d\tilde{\psi}_1 d\tilde{\psi}_2 = \int e^{-S(\tilde{x}, 0, 0)} d\tilde{x} d\tilde{\psi}_1 d\tilde{\psi}_2$$

In our coordinate change, our Lagrangian densities change

$$e^{-\frac{h'(x)^2}{2}} (1 - h''(x)\psi_1\psi_2) \rightarrow e^{-\frac{h'(\bar{x})^2}{2}}$$

One has fermion factors, and the other doesn't. The integral can therefore only be nonzero where the coordinate change can't happen, i.e. when $h'(x) = 0$. We conclude that the integral is only contributed to by critical points of $h(x)$. $\square$

Question 11.6. *Verify that the supersymmetry transformations*

$$\begin{aligned}\delta z &= \epsilon_1 \psi_1 + \epsilon_2 \psi_2 \\ \delta \psi_1 &= \epsilon_2 \overline{W'}(z) \\ \delta \psi_2 &= -\epsilon_1 \overline{W'}(z) \\ \overline{\delta z} &= \epsilon_1 \overline{\psi}_1 + \epsilon_2 \overline{\psi}_2 \\ \overline{\delta \psi}_1 &= \epsilon_2 W'(z) \\ \overline{\delta \psi}_2 &= -\epsilon_1 W'(z)\end{aligned}$$

preserve the action

$$S = |W'(z)|^2 - (W''(z))\psi_1\psi_2 - \left(\overline{W''(z)}\right)\overline{\psi}_1\overline{\psi}_2$$

Recall that ϵ_i are anticommuting parameters.

Proof. Varying different parts of the action, we have

$$\begin{aligned}\delta(|W'(z)|^2) &= \delta(\overline{W'}(z)W'(z)) = \overline{W''(z)}\delta\overline{z}W'(z) + \overline{W'}(z)W''(z)\delta z, \\ \delta(W''(z)\psi_1\psi_2) &= W'''(z)\delta z\psi_1\psi_2 + W''(z)\delta\psi_1\psi_2 + W''(z)\psi_1\delta\psi_2, \\ \delta(\overline{W''(z)}\overline{\psi}_1\overline{\psi}_2) &= \overline{W'''(z)}\delta\overline{z}\overline{\psi}_1\overline{\psi}_2 + \overline{W''(z)}\delta\overline{\psi}_1\overline{\psi}_2 + \overline{W''(z)}\overline{\psi}_1\delta\overline{\psi}_2\end{aligned}$$

Therefore the variation of the action is

$$\begin{aligned}\delta S &= \overline{W''(z)}W'(z)(\epsilon_1\overline{\psi}_1 + \epsilon_2\overline{\psi}_2) + \overline{W'}(z)W''(z)(\epsilon_1\psi_1 + \epsilon_2\psi_2) \\ &\quad - W'''(z)(\epsilon_1\psi_1 + \epsilon_2\psi_2)\psi_1\psi_2 - W''(z)\epsilon_2\overline{W'}(z)\psi_2 + W''(z)\psi_1\epsilon_1\overline{W'}(z) \\ &\quad - \overline{W'''(z)}(\epsilon_1\overline{\psi}_1 + \epsilon_2\overline{\psi}_2)\overline{\psi}_1\overline{\psi}_2 - \overline{W''(z)}\epsilon_2W'(z)\overline{\psi}_2 + \overline{W''(z)}\overline{\psi}_1\epsilon_1W'(z) \\ &= \overline{W''(z)}W'(z)(\epsilon_1\overline{\psi}_1 + \epsilon_2\overline{\psi}_2) + \overline{W'}(z)W''(z)(\epsilon_1\psi_1 + \epsilon_2\psi_2) \\ &\quad - W''(z)\epsilon_2\overline{W'}(z)\psi_2 + W''(z)\psi_1\epsilon_1\overline{W'}(z) \\ &\quad - \overline{W''(z)}\epsilon_2W'(z)\overline{\psi}_2 + \overline{W''(z)}\overline{\psi}_1\epsilon_1W'(z) \\ &= 0\end{aligned}$$

Therefore these supersymmetric transformations preserve the action. $\square$

Question 11.7. Complete the calculation at the end of this chapter to show that if $W(z)$ is a polynomial and the critical points of $W(z)$ all have multiplicity 1 as solutions of $W'(z) = 0$, then Z , appropriately normalized, is just the number of critical points of W , i.e., the degree of W' .

Proof. With the action again given by

$$S = |W'(z)|^2 - (W''(z))\psi_1\psi_2 - \left(\overline{W''(z)}\right)\bar{\psi}_1\bar{\psi}_2$$

we seek to compute the partition function

$$Z = \frac{1}{2\pi} \int e^{-S} dz d\bar{z} d\psi_1 d\psi_2 d\bar{\psi}_1 d\bar{\psi}_2$$

We start by expanding e^{-S} :

$$\begin{aligned} Z &= \frac{1}{2\pi} \int e^{-|W'(z)|^2} (1 + W''(z)\psi_1\psi_2)(1 + \overline{W''(z)}\bar{\psi}_1\bar{\psi}_2) dz d\bar{z} d\psi_1 d\psi_2 d\bar{\psi}_1 d\bar{\psi}_2 \\ &= \frac{1}{2\pi} \int e^{-|W'(z)|^2} W''(z)\overline{W''(z)} dz d\bar{z} \\ &= \frac{1}{2\pi} \int e^{-W'(z)\overline{W'(z)}} W''(z)\overline{W''(z)} dz d\bar{z} \end{aligned}$$

where we did the Berezin integrals over our fermions $\psi_i, \bar{\psi}_i$.

We examine the partition function near critical points of $W(z)$. Namely, we expand around each critical point z_c :

$$\begin{aligned} W'(z) &\approx \cancel{W'(z_c)} + W''(z_c)(z - z_c) \\ \overline{W'(z)} &\approx \cancel{\overline{W'(z_c)}} + \overline{W''(z_c)}(\bar{z} - \bar{z}_c) \end{aligned}$$

so our partition function becomes

$$\begin{aligned} Z &= \frac{1}{2\pi} \int e^{-W''(z_c)(z-z_c)\overline{W''(z_c)}(\bar{z}-\bar{z}_c)} W''(z_c)\overline{W''(z_c)} dz d\bar{z} \\ &= \frac{|W''(z_c)|^2}{2\pi} \int e^{-|W''(z_c)|^2|z-z_c|^2} dz d\bar{z} \end{aligned}$$

First we simplify our measure. Define $a + ib := z$ for $a, b \in \mathbb{R}$. Recall that $dz d\bar{z}$ is $dz \wedge d\bar{z}$, so we have

$$\begin{aligned} dz \wedge d\bar{z} &= d(a + ib) \wedge d(a - ib) \\ &= da \wedge da - ida \wedge db + idb \wedge da + db \wedge db \\ &= 2idb \wedge da \end{aligned}$$

due to anticommutativity for the wedge product. Our integral then becomes

$$Z = \frac{|W''(z_c)|^2}{2\pi} \int e^{-|W''(z_c)|^2(a^2+b^2)} 2i db da$$

since we're integrating over all space, so we substitute $a^2 + b^2$ for $|z - z_c|^2$ since a and b are our integration variables.

Recall that a Gaussian integral evaluates $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$, so our partition function integrates to

$$Z = \frac{|W''(z_c)|^2}{2\pi} 2i \left(\sqrt{\frac{\pi}{|W''(z_c)|^2}} \right)^2 = i$$

around each zero z_c . Thus the partition function counts the number of critical points of W , or, if the critical points of $W(z)$ all have multiplicity 1, the degree of W' . $\square$

12 Introduction to String Theory

Question 12.1. Suppose that a compact orientable C^∞ surface Σ is equipped with a Riemannian metric.

- a Pick a chart with coordinates (x_1, x_2) and an overlapping chart with coordinates (y_1, y_2) , so that y_1 and y_2 can be expressed as functions of x_1 and x_2 . If the metric is written as $\sum_{ij} g_{ij}(x) dx_i \otimes dx_j$ in the first chart, find the expression $\sum_{ij} h_{ij}(y) dy_i \otimes dy_j$ for the same metric in the second chart.
- b Verify that the condition of the metric being positive definite is independent of coordinates, as claimed.
- c Verify that the area form, interpreted as a 2-form, is independent of coordinates, as claimed.

Proof. a Recall that the cotangent coordinates are such that $dx_j = \sum_k \frac{\partial x_j}{\partial y_k} dy_k$.

Then the metric is written

$$\begin{aligned}\sum_{ij} g_{ij}(x) dx_i \otimes dx_j &= \sum_{ij} g_{ij}(x) \sum_k \frac{\partial x_i}{\partial y_k} dy_k \otimes \sum_m \frac{\partial x_j}{\partial y_m} dy_m \\ &= \sum_{ij} g_{ij}(x) \sum_{k,m} \frac{\partial x_i}{\partial y_k} \frac{\partial x_j}{\partial y_m} dy_k \otimes dy_m\end{aligned}$$

b Recall that, similar to cotangent coordinates, we have $\frac{\partial}{\partial x_i} = \sum_j \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}$. If

$\sum_{ij} g_{ij}(x) dx_i \otimes dx_j$ is positive definite, then the two eigenvalues of the metric are positive throughout the coordinates (x_1, x_2) .

Let $(a_1(x) \frac{\partial}{\partial x_1}, a_2(x) \frac{\partial}{\partial x_2})^T$ be eigenvectors of g_{ij} . We thus have

$$g_{ij} \begin{pmatrix} a_1(x) \\ a_2(x) \end{pmatrix} = \begin{pmatrix} \lambda_1(x) a_1(x) \\ \lambda_2(x) a_2(x) \end{pmatrix}$$

Changing bases to our (y_1, y_2) coordinate basis, we have

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_1} a_1(x) + \frac{\partial y_1}{\partial x_2} a_2(x) \\ \frac{\partial y_2}{\partial x_1} a_1(x) + \frac{\partial y_2}{\partial x_2} a_2(x) \end{pmatrix}$$

This gives our change of basis matrix:

$$A = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{pmatrix}$$

Note that $\det(A) = \frac{\partial y_1}{\partial x_1} \frac{\partial y_2}{\partial x_2} - \frac{\partial y_2}{\partial x_1} \frac{\partial y_1}{\partial x_2} \neq 0$, since the coordinates must span 2 dimensions, i.e. x_1, x_2 must be linearly independent, as must y_1, y_2 . Therefore A is invertible, and we note that $h_{ij} = A^{-1} g_{ij} A$. If $\mathbf{x}$ is an eigenvector of g_{ij} , we have $g_{ij} \mathbf{x} = \lambda \mathbf{x}$. Thus we have

$$\begin{aligned}\mathbf{x} &= A^{-1} \mathbf{y} \Rightarrow \\ g_{ij} \mathbf{x} &= g_{ij} A^{-1} \mathbf{y} \\ &= \lambda A^{-1} \mathbf{y} \Rightarrow \\ A g_{ij} \mathbf{x} &= A g_{ij} A^{-1} \mathbf{y} = h_{ij} \mathbf{y} = \lambda h_{ij} \mathbf{y} = \lambda \mathbf{y}\end{aligned}$$

Thus the eigenvalues are coordinate-independent, and if they are positive, the metric remains positive-definite independent of coordinates.

- c Since the eigenvalues are basis independent, we can express our metric in the diagonal basis

$$\begin{pmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{pmatrix}$$

Therefore $\det(g(x)) = \lambda_1(x)\lambda_2(x)$, which is always the same value, independent of coordinates, and thus the area form $\Phi = \sqrt{\det g(x)} dx_1 \wedge dx_2$ is coordinate-independent. $\square$

Question 12.2. Let M be a complex manifold, and let TM denote its real tangent bundle when M is viewed as a real manifold.

- a Prove that $TM \otimes \mathbb{C}$ is isomorphic to $T^{1,0}M \oplus T^{0,1}M$ as complex vector bundles on M .

- b Prove that $T^{1,0}M$ is a holomorphic vector bundle on M .

Proof. First let f be a change of coordinates on M such that $f(\mathbf{z}') = \mathbf{z}$. Let $f(\mathbf{z}') = f(\mathbf{x}' + i\mathbf{y}') := \sum_j x_j(\mathbf{x}', \mathbf{y}') + i \sum_j y_j(\mathbf{x}', \mathbf{y}')$, where $\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}'$ are real coordinates on M . If f is holomorphic, then the Cauchy-Riemann equations state that $\frac{\partial f}{\partial \bar{z}_j} = 0, \forall j = 1, \dots, n$. It follows that

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}_j} &:= \frac{\partial \mathbf{z}}{\partial \bar{z}_j} \\ &= \sum_i \frac{1}{2} \left(\frac{\partial x_i}{\partial x'_j} + i \frac{\partial x_i}{\partial y'_j} + i \frac{\partial y_i}{\partial x'_j} - \frac{\partial y_i}{\partial y'_j} \right) = 0 \\ &\Rightarrow \frac{\partial x_i}{\partial x'_j} = \frac{\partial y_i}{\partial y'_j}, \quad \frac{\partial x_i}{\partial y'_j} = -\frac{\partial y_i}{\partial x'_j} \end{aligned}$$

for all $i, j = 1, \dots, n$.

- a Recall that an arbitrary vector in $T_x M \otimes \mathbb{C}$ is given by

$$\sum_{i=1}^n (a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial y_i}), \{a_i, b_i\} \in \mathbb{C}$$

with transition functions to coordinates (x', y') given by

$$\begin{pmatrix} \frac{\partial x_1}{\partial x'_1} & \cdots & \frac{\partial x_n}{\partial x'_1} & \frac{\partial y_1}{\partial x'_1} & \cdots & \frac{\partial y_n}{\partial x'_1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial x'_n} & \cdots & \frac{\partial x_n}{\partial x'_n} & \frac{\partial y_1}{\partial x'_n} & \cdots & \frac{\partial y_n}{\partial x'_n} \\ \frac{\partial x_1}{\partial y'_1} & \cdots & \frac{\partial x_n}{\partial y'_1} & \frac{\partial y_1}{\partial y'_1} & \cdots & \frac{\partial y_n}{\partial y'_1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial y'_n} & \cdots & \frac{\partial x_n}{\partial y'_n} & \frac{\partial y_1}{\partial y'_n} & \cdots & \frac{\partial y_n}{\partial y'_n} \end{pmatrix} = \begin{pmatrix} D_{\mathbf{x}'\mathbf{x}} & D_{\mathbf{x}'\mathbf{y}} \\ D_{\mathbf{y}'\mathbf{x}} & D_{\mathbf{y}'\mathbf{y}} \end{pmatrix} = \begin{pmatrix} D_{\mathbf{x}'\mathbf{x}} & D_{\mathbf{x}'\mathbf{y}} \\ -D_{\mathbf{x}'\mathbf{y}} & D_{\mathbf{x}'\mathbf{x}} \end{pmatrix}$$

where $D_{\cdot'}$ is the Jacobian with respect to the $\cdot'$ variables, and the second equality is due to the Cauchy-Riemann equations. Furthermore, an arbitrary vector in $T_x^{1,0}M \oplus T_x^{0,1}M$ is given by

$$\left(\sum_{i=1}^n c_i \frac{1}{2} \left(\frac{\partial}{\partial x_i} - i \frac{\partial}{\partial y_i} \right), \sum_{i=1}^n d_i \frac{1}{2} \left(\frac{\partial}{\partial x_i} + i \frac{\partial}{\partial y_i} \right) \right)$$

The transition functions for $T^{1,0}M$ are given by

$$g_{ij}^{1,0} := \begin{pmatrix} \frac{\partial z_1}{\partial z'_1} & \cdots & \frac{\partial z_n}{\partial z'_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial z_1}{\partial z'_n} & \cdots & \frac{\partial z_n}{\partial z'_n} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \left(\frac{\partial x_1}{\partial x'_1} + \frac{\partial y_1}{\partial y'_1} - i \left(\frac{\partial x_1}{\partial y'_1} - \frac{\partial y_1}{\partial x'_1} \right) \right) & \cdots & \frac{1}{2} \left(\frac{\partial x_n}{\partial x'_1} + \frac{\partial y_n}{\partial y'_1} - i \left(\frac{\partial x_n}{\partial y'_1} - \frac{\partial y_n}{\partial x'_1} \right) \right) \\ \vdots & \ddots & \vdots \\ \frac{1}{2} \left(\frac{\partial x_1}{\partial x'_n} + \frac{\partial y_1}{\partial y'_n} - i \left(\frac{\partial x_1}{\partial y'_n} - \frac{\partial y_1}{\partial x'_n} \right) \right) & \cdots & \frac{1}{2} \left(\frac{\partial x_n}{\partial x'_n} + \frac{\partial y_n}{\partial y'_n} - i \left(\frac{\partial x_n}{\partial y'_n} - \frac{\partial y_n}{\partial x'_n} \right) \right) \end{pmatrix} = \frac{1}{2} (D_{\mathbf{x}'\mathbf{x}} + D_{\mathbf{y}'\mathbf{y}} - i D_{\mathbf{y}'\mathbf{x}} + i D_{\mathbf{x}'\mathbf{y}})$$

and the transition functions for $T^{0,1}M$ are given by

$$g_{ij}^{0,1} := \begin{pmatrix} \frac{\partial \bar{z}_1}{\partial \bar{z}_1'} & \dots & \frac{\partial \bar{z}_n}{\partial \bar{z}_1'} \\ \vdots & \ddots & \vdots \\ \frac{\partial \bar{z}_1}{\partial \bar{z}_n'} & \dots & \frac{\partial \bar{z}_n}{\partial \bar{z}_n'} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{1}{2} \left(\frac{\partial x_1}{\partial x_1'} + \frac{\partial y_1}{\partial y_1'} + i \left(\frac{\partial x_1}{\partial y_1'} - \frac{\partial y_1}{\partial x_1'} \right) \right) & \dots & \frac{1}{2} \left(\frac{\partial x_n}{\partial x_1'} + \frac{\partial y_n}{\partial y_1'} + i \left(\frac{\partial x_n}{\partial y_1'} - \frac{\partial y_n}{\partial x_1'} \right) \right) \\ \vdots & \ddots & \vdots \\ \frac{1}{2} \left(\frac{\partial x_1}{\partial x_n'} + \frac{\partial y_1}{\partial y_n'} + i \left(\frac{\partial x_1}{\partial y_n'} - \frac{\partial y_1}{\partial x_n'} \right) \right) & \dots & \frac{1}{2} \left(\frac{\partial x_n}{\partial x_n'} + \frac{\partial y_n}{\partial y_n'} + i \left(\frac{\partial x_n}{\partial y_n'} - \frac{\partial y_n}{\partial x_n'} \right) \right) \end{pmatrix} =$$

$$\frac{1}{2} (D_{\mathbf{x}'} \mathbf{x} + D_{\mathbf{y}'} \mathbf{y} + i D_{\mathbf{y}'} \mathbf{x} - i D_{\mathbf{x}'} \mathbf{y})$$

Thus the transition functions for $T^{1,0}M \oplus T^{0,1}M$ are

$$\begin{pmatrix} \frac{1}{2} (D_{\mathbf{x}'} \mathbf{x} + D_{\mathbf{y}'} \mathbf{y} - i D_{\mathbf{y}'} \mathbf{x} + i D_{\mathbf{x}'} \mathbf{y}) & 0 \\ 0 & \frac{1}{2} (D_{\mathbf{x}'} \mathbf{x} + D_{\mathbf{y}'} \mathbf{y} - i D_{\mathbf{y}'} \mathbf{x} + i D_{\mathbf{x}'} \mathbf{y}) \end{pmatrix}$$

Applying the Cauchy-Riemann equations, we then seek to find a matrix P such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} D_{\mathbf{x}'} \mathbf{x} + i D_{\mathbf{x}'} \mathbf{y} & 0 \\ 0 & D_{\mathbf{x}'} \mathbf{x} - i D_{\mathbf{x}'} \mathbf{y} \end{pmatrix}$$

$$= \begin{pmatrix} D_{\mathbf{x}'} \mathbf{x} & D_{\mathbf{x}'} \mathbf{y} \\ -D_{\mathbf{x}'} \mathbf{y} & D_{\mathbf{x}'} \mathbf{x} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Thus we have

$$\begin{pmatrix} A D_{\mathbf{x}'} \mathbf{x} + i A D_{\mathbf{x}'} \mathbf{y} & B D_{\mathbf{x}'} \mathbf{x} - i B D_{\mathbf{x}'} \mathbf{y} \\ C D_{\mathbf{x}'} \mathbf{x} + i C D_{\mathbf{x}'} \mathbf{y} & D D_{\mathbf{x}'} \mathbf{x} - i D D_{\mathbf{x}'} \mathbf{y} \end{pmatrix} =$$

$$\begin{pmatrix} D_{\mathbf{x}'} \mathbf{x} A + D_{\mathbf{x}'} \mathbf{y} C & D_{\mathbf{x}'} \mathbf{x} B + D_{\mathbf{x}'} \mathbf{y} D \\ D_{\mathbf{x}'} \mathbf{x} C - D_{\mathbf{x}'} \mathbf{y} A & D_{\mathbf{x}'} \mathbf{x} D - D_{\mathbf{x}'} \mathbf{y} B \end{pmatrix}$$

Thus we require the sub-matrices in P to be diagonal, and $C = iA$ and $D = iB$, so we have

$$P = \begin{pmatrix} A & iD \\ iA & D \end{pmatrix}$$

To have P be invertible, we must require A and D to be invertible:

$$P^{-1} = \begin{pmatrix} \frac{1}{2}A^{-1} & -\frac{i}{2}A^{-1} \\ -\frac{i}{2}D^{-1} & \frac{1}{2}D^{-1} \end{pmatrix}$$

This is easy to do, (let $A = D = Id_{n \times n}$ to be concrete), so these bundles are isomorphic.

- b It is given in the book that the fibers of $T_x^{1,0}M$ are spanned by $\{\frac{\partial}{\partial z_j}\}$, where $(z_1, \dots, z_n)$ are any holomorphic coordinates in any chart on M containing x . The transition functions for $T^{1,0}M$ are the Jacobians $\frac{\partial z_i}{\partial z'_j}$. Since z_i are holomorphic functions, their derivatives are also holomorphic, so the transition functions are also holomorphic.

□

Question 12.3. *Show that the bosonic string action*

$$S(f, g) = -\frac{1}{2\pi\alpha'} \int_{\Sigma} \sum_{i=1}^n \sum_{j,k=1}^2 \left(g^{jk}(x) \frac{\partial f_i}{\partial x_j} \frac{\partial f_i}{\partial x_k} \right) \Phi,$$

where $f : \Sigma \rightarrow \mathbb{R}^n$ is a C^∞ map, (Σ is a Riemannian surface with metric g_{ij} with inverse g^{ij}), is independent of the coordinates that it is expressed in, so that the bosonic string action makes sense.

Proof. In the first problem of the chapter, we showed that the area form Φ is coordinate-independent. In a chart with coordinates $\mathbf{x}$, we rewrite the integrand in matrix form:

$$\sum_{i=1}^n (\partial_{\mathbf{x}} f_i)^T g_{jk}^{-1} (\partial_{\mathbf{x}} f_i)$$

Note that $\partial_{\mathbf{x}} f_i$ are covectors, as $\sum \frac{\partial f_i}{\partial x_j} dx_j$ gives the differential form df .

The metric is written

$$\sum_{i,j=1}^2 g_{ij}(x) dx_i \otimes dx_j$$

Since differential forms change via $dx'_i = \sum \frac{\partial x'_i}{\partial x_j} dx_j$, if this chart intersects with another chart that has coordinates $\mathbf{x}'$, the metric transforms

$$\begin{aligned} g_{ij}(\mathbf{x}) dx_i \otimes dx_j &\mapsto \sum_{i,j=1}^2 g_{ij} \left(\sum_{k=1}^2 \frac{\partial x'_i}{\partial x_k} dx_k \otimes \sum_{k=1}^2 \frac{\partial x'_j}{\partial x_k} dx_k \right) \\ &= (D_{\mathbf{x}'} \mathbf{x})^{-T} g_{ij}(\mathbf{x}) (D_{\mathbf{x}'} \mathbf{x})^{-1} \\ &\Rightarrow g^{ij} = g_{ij}^{-1} \mapsto D_{\mathbf{x}'} \mathbf{x} g^{ij} (D_{\mathbf{x}'} \mathbf{x})^T \end{aligned}$$

Furthermore, the covectors transform as

$$\begin{aligned} \frac{\partial f_i}{\partial x_j} &\mapsto \sum \frac{\partial x_k}{\partial x'_j} \frac{\partial f_i}{\partial x_k} \\ &= (D_{\mathbf{x}'} \mathbf{x})^{-T} \frac{\partial f_i}{\partial x_j} \end{aligned}$$

Thus under a change of coordinates the integrand becomes

$$\begin{aligned} \sum_{i=1}^n ((D_{\mathbf{x}'} \mathbf{x})^{-T} \partial_{\mathbf{x}} f_i)^T (D_{\mathbf{x}'} \mathbf{x} g^{ij} (D_{\mathbf{x}'} \mathbf{x})^T) (D_{\mathbf{x}'} \mathbf{x})^{-T} \partial_{\mathbf{x}} f_i \\ = \sum_{i=1}^n (\partial_{\mathbf{x}} f_i)^T g^{jk} (\partial_{\mathbf{x}} f_i) \end{aligned}$$

which is the same integrand. Thus the integral is coordinate-independent, so the bosonic string action is coordinate-independent. $\square$

Question 12.4. *Show that the term*

$$\frac{1}{2\pi\alpha'} \int_{\Sigma} \left(\sum_{i,j=1}^n h_{i\bar{j}}(f(z)) \left(\psi_z^{\bar{j}} \frac{\partial \chi^i}{\partial \bar{z}} + \psi_{\bar{z}}^i \frac{\partial \chi^{\bar{j}}}{\partial z} \right) \right) dz \wedge d\bar{z}$$

in the A-model action is independent of the choice of coordinates in which it is expressed.

Proof. Recall that f is a map $f : \Sigma \rightarrow X$, $h_{i\bar{j}}$ is the Kähler metric on X ,

- $\chi(z)$ is a C^∞ section of $f^*TX \otimes \mathbb{C}$

- $\psi_z(z)$ is a C^∞ section of $(T^{1,0}\Sigma)^* \otimes f^*T^{0,1}X$
- $\psi_{\bar{z}}(z)$ is a C^∞ section of $(T^{0,1}\Sigma)^* \otimes f^*T^{1,0}X$

When changing coordinates on X from $(w_1, \bar{w}_1, \dots, w_n, \bar{w}_n) := \mathbf{w}$ to $\mathbf{w}'$, we have from before that $h_{i\bar{j}}(f(z')) = (D_{\mathbf{w}'}\mathbf{w})^{-T} h_{i\bar{j}}(f(z))(D_{\mathbf{w}'}\mathbf{w})^{-1}$.

We know from the previous problem that vector fields transform via $(D_{\mathbf{w}'}\mathbf{w})\chi(z)$, and that covectors transform via $(D_{\mathbf{x}'}\mathbf{x})^{-T}\psi_z(z)$.

Thus under a change of coordinates on $X = \mathbb{C}^n$, we get

$$\begin{aligned} & \left(\psi_z^{\bar{j}} dz \otimes \left((D_{\mathbf{w}'}\mathbf{w}) f^* \left(\frac{\partial}{\partial \bar{z}_i} \right) \right)^T \right) (D_{\mathbf{w}'}\mathbf{w})^{-T} h_{i\bar{j}}(f(z))(D_{\mathbf{w}'}\mathbf{w})^{-1} \left(D_{\mathbf{w}'}\mathbf{w} \frac{\partial \chi^j}{\partial \bar{z}} \right) + \\ & \left(\psi_{\bar{z}}^i d\bar{z} \otimes \left((D_{\mathbf{w}'}\mathbf{w}) f^* \left(\frac{\partial}{\partial \bar{z}_i} \right) \right)^T \right) (D_{\mathbf{w}'}\mathbf{w})^{-T} h_{i\bar{j}}(f(z))(D_{\mathbf{w}'}\mathbf{w})^{-1} \left(D_{\mathbf{w}'}\mathbf{w} \frac{\partial \chi^{\bar{j}}}{\partial z} \right) \\ & = \psi_z^{\bar{j}} dz \otimes f^* \left(\frac{\partial}{\partial \bar{z}_i} \right) h_{i\bar{j}}(f(z)) \frac{\partial \chi^j}{\partial \bar{z}} + \psi_{\bar{z}}^i d\bar{z} \otimes f^* \left(\frac{\partial}{\partial \bar{z}_i} \right) h_{i\bar{j}}(f(z)) \frac{\partial \chi^{\bar{j}}}{\partial z} \end{aligned}$$

which is our same action as before.

For a change of coordinates on Σ defined by $(z, \bar{z}) \mapsto (z', \bar{z}')$, we have

$$dz \wedge d\bar{z} \mapsto \left(\frac{\partial z}{\partial z'} \frac{\partial \bar{z}}{\partial \bar{z}'} - \frac{\partial \bar{z}}{\partial z'} \frac{\partial z}{\partial \bar{z}'} \right) dz' \wedge d\bar{z}'$$

For a holomorphic coordinate change, the Cauchy-Riemann equations require

$$\frac{\partial \bar{z}}{\partial z'} = \frac{\partial z}{\partial \bar{z}'} = 0$$

so we get that $dz' \wedge d\bar{z}' = \left| \frac{\partial z}{\partial z'} \right|^{-2} dz \wedge d\bar{z}$. Furthermore, we have

$$\begin{aligned} \psi_z^{\bar{j}} \frac{\partial \chi^i}{\partial \bar{z}} & \mapsto \psi_z^{\bar{j}} \left(\frac{\partial z}{\partial z'} + \frac{\partial z}{\partial \bar{z}'} \right) \frac{\partial \chi^i}{\partial \bar{z}} \left(\frac{\partial \bar{z}}{\partial z'} + \frac{\partial \bar{z}}{\partial \bar{z}'} \right) \\ \psi_{\bar{z}}^i \frac{\partial \chi^{\bar{j}}}{\partial z} & \mapsto \psi_{\bar{z}}^i \left(\frac{\partial \bar{z}}{\partial z'} + \frac{\partial \bar{z}}{\partial \bar{z}'} \right) \frac{\partial \chi^{\bar{j}}}{\partial z} \left(\frac{\partial z}{\partial z'} + \frac{\partial z}{\partial \bar{z}'} \right) \end{aligned}$$

In order to keep $\psi_z^{\bar{j}}$ as a section of $(T^{1,0}\Sigma)^*$ and $\psi_{\bar{z}}^i$ as a section of $(T^{0,1}\Sigma)^*$, along with the correct derivatives of χ components, this actually is

$$\begin{aligned}\psi_z^{\bar{j}} \frac{\partial \chi^i}{\partial \bar{z}} &\mapsto \psi_z^{\bar{j}} \left(\frac{\partial z}{\partial z'} \right) \frac{\partial \chi^i}{\partial \bar{z}} \left(\frac{\partial \bar{z}}{\partial \bar{z}'} \right) = \psi_z^{\bar{j}} \frac{\partial \chi^i}{\partial \bar{z}} \left| \frac{\partial z}{\partial z'} \right|^2 \\ \psi_{\bar{z}}^i \frac{\partial \chi^{\bar{j}}}{\partial z} &\mapsto \psi_{\bar{z}}^i \left(\frac{\partial \bar{z}}{\partial \bar{z}'} \right) \frac{\partial \chi^{\bar{j}}}{\partial z} \left(\frac{\partial z}{\partial z'} \right) = \psi_{\bar{z}}^i \frac{\partial \chi^{\bar{j}}}{\partial z} \left| \frac{\partial z}{\partial z'} \right|^2\end{aligned}$$

Thus we have

$$\begin{aligned}\frac{1}{2\pi\alpha'} \int_{\Sigma} \left(\sum_{i,j=1}^n h_{i\bar{j}}(f(z)) \left(\psi_z^{\bar{j}} \frac{\partial \chi^i}{\partial \bar{z}} + \psi_{\bar{z}}^i \frac{\partial \chi^{\bar{j}}}{\partial z} \right) \right) dz \wedge d\bar{z} &\mapsto \\ \frac{1}{2\pi\alpha'} \int_{\Sigma} \left(\sum_{i,j=1}^n h_{i\bar{j}}(f(z')) \left(\psi_{z'}^{\bar{j}} \frac{\partial \chi^i}{\partial \bar{z}'} \left| \frac{\partial z}{\partial z'} \right|^2 + \psi_{\bar{z}'}^i \frac{\partial \chi^{\bar{j}}}{\partial z'} \left| \frac{\partial z}{\partial z'} \right|^2 \right) \right) \left| \frac{\partial z}{\partial z'} \right|^{-2} dz' \wedge d\bar{z}' & \\ \frac{1}{2\pi\alpha'} \int_{\Sigma} \left(\sum_{i,j=1}^n h_{i\bar{j}}(f(z')) \left(\psi_{z'}^{\bar{j}} \frac{\partial \chi^i}{\partial \bar{z}'} + \psi_{\bar{z}'}^i \frac{\partial \chi^{\bar{j}}}{\partial z'} \right) \right) dz' \wedge d\bar{z}' &\end{aligned}$$

Thus the action is unchanged under holomorphic change of coordinates. $\square$

Question 12.5. *This problem considers transformations T of fields*

$$F = \sum_{k=1}^{2n} \sum_{J^k=(j_1^k < \dots < j_k^k)} f_{J^k}(x^1, \dots, x^{2n}) \chi^{j_1^k} \dots \chi^{j_k^k}$$

to differential forms on X :

$$T(F) = \sum_{k=1}^{2n} \sum_{J^k=(j_1^k < \dots < j_k^k)} f_{J^k}(x_1, \dots, x_{2n}) dx_{j_1^k} \wedge \dots \wedge dx_{j_k^k}$$

- Show that the transformation from fields to differential forms defined in coordinates above is independent of coordinates.
- Show that under this transformation, the BRST operator Q corresponds to the usual exterior derivative d .

Proof. a Under a change of coordinates $x^I \mapsto y^I$ on X , we know from before that $\chi^i \mapsto \chi^i \sum_j \frac{\partial y_j}{\partial x_i}$ since χ^i is a section of the tangent bundle.

Our χ^i are fermions, since we have $T(\chi^i \chi^j) = dx_i \wedge dx_j = -dx_j \wedge dx_i = -T(\chi^j \chi^i)$. The linearity of the wedge product applies to our χ^i in the same way. Therefore a coordinate transformation gives

$$\begin{aligned} \sum_{J^k=(j_1^k < \dots < j_k^k)} f_{J^k}(x^1, \dots, x^{2n}) \chi^{j_1^k} \dots \chi^{j_k^k} &\mapsto \\ \sum_{J^k=(j_1^k < \dots < j_k^k)} f_{J^k}(x^1, \dots, x^{2n}) \det(D_{\mathbf{x}} \mathbf{y}) \chi^{j_1^k} \dots \chi^{j_k^k} \end{aligned}$$

where $D_{\mathbf{x}} \mathbf{y}$ is the $k \times k$ Jacobian of the coordinate change. We know furthermore from before that the wedge product of the differential forms transforms under changes of coordinates in exactly the same way:

$$\begin{aligned} \sum_{J^k=(j_1^k < \dots < j_k^k)} f_{J^k}(x_1, \dots, x_{2n}) dx_{j_1^k} \wedge \dots \wedge dx_{j_k^k} &\mapsto \\ \sum_{J^k=(j_1^k < \dots < j_k^k)} f_{J^k}(x_1, \dots, x_{2n}) \det(D_{\mathbf{x}} \mathbf{y}) dx_{j_1^k} \wedge \dots \wedge dx_{j_k^k} \end{aligned}$$

Thus the diagram commutes:

$$\begin{array}{ccc} F(\mathbf{x}) & \longrightarrow & F(\mathbf{y}) \\ \downarrow & & \downarrow \\ T(F(\mathbf{x})) & \longrightarrow & T(F(\mathbf{y})) \end{array}$$

b We examine $Q(g(x^i)\chi^j)$:

$$\begin{aligned}
Q(g(x^i)\chi^j) &= Q(g(x^i))\chi^j + g(x^i)Q(\chi^j) \\
&= Q(g(x^i))\chi^j \\
&\approx Q(g(a) + \frac{\partial g}{\partial x^i}(a)(x^i - a))\chi^j \\
&= [Q(g(a)) + Q(\frac{\partial g}{\partial x^i}(a))(x^i - a) + \frac{\partial g}{\partial x^i}(a)Q(x^i - a)]\chi^j \\
&= \frac{\partial g}{\partial x^i}(a)Q(x^i)\chi^j \\
&= \frac{\partial g}{\partial x^i}\chi^i\chi^j
\end{aligned}$$

where we used that Q is a derivation, $Q(\chi^I) = 0$, Taylor expansion, the fact that Q being a derivation implies that Q of a constant is 0, and $Q(x^I) = \chi^I$.

Thus, for general functions F , we have

$$\begin{aligned}
Q(F) &:= Q\left(\sum_{k=1}^{2n} \sum_{J^k=(j_1^k < \dots < j_k^k)} f_{J^k}(x^1, \dots, x^{2n}) \chi^{j_1^k} \dots \chi^{j_k^k}\right) \\
&= \sum_{k=1}^{2n} \sum_{J^k=(j_1^k < \dots < j_k^k)} \sum_{m=1}^{2n} \frac{\partial f_{J^k}}{\partial x^m}(x^1, \dots, x^{2n}) \chi^m \chi^{j_1^k} \dots \chi^{j_k^k} \Rightarrow \\
T(Q(F)) &= \sum_{k=1}^{2n} \sum_{J^k=(j_1^k < \dots < j_k^k)} \sum_{m=1}^{2n} \frac{\partial f_{J^k}}{\partial x^m}(x_1, \dots, x_{2n}) dx_m \wedge dx_{j_1^k} \wedge \dots \wedge dx_{j_k^k}
\end{aligned}$$

This is exactly the formula of the exterior derivative $dT(F)$.

□

13 Topological Quantum Field Theory

Question 13.1. Show that any holomorphic section of the canonical bundle of $\mathbb{P}^1$ is identically zero.

Proof. We consider the coordinates according to the usual cover of $\mathbb{P}^n$: $U_0 = \{(x_0, x_1) \in \mathbb{P}^1 | x_0 \neq 0\}$, $U_1 = \{(x_0, x_1) \in \mathbb{P}^1 | x_1 \neq 0\}$ with coordinates $y := \frac{x_1}{x_0}$

and $z := \frac{x_0}{x_1}$. The transition functions for the canonical bundle are $g_{zy} = \frac{\partial z}{\partial y}$ and $g_{yz} = \frac{\partial y}{\partial z}$. For the first one, substituting $\frac{1}{y}$ for z , we have $g_{zy} = -\frac{1}{y^2}$. Thus our holomorphic section ω must satisfy $\omega(z) = -\frac{x_0^2}{x_1^2}\omega(y)$ whenever $x_0, x_1 \neq 0$. Because ω is holomorphic, for each point in $U_1 \cap U_2$, this equality can be re-expressed as convergent power series:

$$\sum_{i=0}^{\infty} a_i \left(\frac{x_0}{x_1} - a \right)^i = -\frac{x_0^2}{x_1^2} \sum_{i=0}^{\infty} b_i \left(\frac{x_1}{x_0} - a \right)^i$$

No matter which i term is considered, there is always a mismatch in powers on either side, since a cannot be zero ($0 \notin \mathbb{P}^1$). Thus the only $\{a_i\}$ and $\{b_i\}$ that satisfy this for any $a \in U_1 \cap U_2$ are such that all values are identically 0, i.e. ω must be zero. $\square$

Question 13.2. *Given a complex manifold X of dimension n , a holomorphic n -form ω on X , and a point $x \in X$, show that the notion given in the text for ω to vanish at x ($\omega(z_1(x), \dots, z_n(x)) = 0$ where $(z_1(x), \dots, z_n(x))$ are the coordinates of x) when expressed in a given coordinate system is in fact independent of the choice of coordinates.*

Proof. Note that, in the first coordinate system, ω is expressed as

$$\omega(z_1(x), \dots, z_n(x)) dz_1 \wedge \dots \wedge dz_n$$

In different coordinates $z'_1(x), \dots, z'_n(x)$, we have

$$\omega(z_1(x), \dots, z_n(x)) dz_1 \wedge \dots \wedge dz_n = \omega(z_1(x), \dots, z_n(x)) \det \left(\frac{z'_j}{z_i} \right) dz'_1 \wedge \dots \wedge dz'_n$$

If $\omega(z_1(x), \dots, z_n(x)) = 0$, then $\omega(z_1(x), \dots, z_n(x)) \det \left(\frac{z'_j}{z_i} \right) = 0$ as well, and thus the zero of this n -form is independent of coordinates. $\square$

Question 13.3. *Let X be a compact complex manifold of dimension n . Show that X is Calabi-Yau if and only if the canonical bundle of X is isomorphic to the trivial line bundle.*

Proof. Suppose the canonical bundle of X is isomorphic to the trivial line bundle, i.e. $X \times \mathbb{C}$. Then, for $a \neq 0 \in \mathbb{C}$, the section $X \times a(x)$ of the trivial bundle is nowhere vanishing. If the bundles are isomorphic, there exist nowhere vanishing holomorphic functions f_α on each chart of X such that $g_{triv} = f_\alpha g_{\alpha\beta} f_\beta^{-1}$ for all transition functions $g_{\alpha\beta}$ of the canonical bundle. Since g_{triv} , $g_{\alpha\beta}$, and all f_α are nonvanishing, and the section $X \times \{a\}$ is nonvanishing, there is an associated nonvanishing section of the canonical bundle, and X must be Calabi-Yau.

Now suppose that X is Calabi-Yau. Label the associated nonvanishing n -form Ω . Now suppose we have any section of the trivial line bundle, labeled $X \times a(x)$. The map $X \times \mathbb{C} \rightarrow \wedge^n T^*X$ given by $a \mapsto a\Omega$ is invertible: given a section Ω' of $\wedge^n T^*X$, we get $\frac{\Omega'(x)}{\Omega(x)}$ is a complex number associated to each point $x \in X$, i.e. a section of $X \times \mathbb{C}$. Because Ω is nonvanishing, this map is always defined - thus the bundles are isomorphic. $\square$

Question 13.4. Let Y_ψ be the family of quintic hypersurfaces in $\mathbb{P}^4$ defined by the equations

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0$$

and G be the finite group

$$G = \{(\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4) \in \mathbb{C}^5 \mid \zeta_i^5 = 1, \prod_{i=0}^4 \zeta_i = 1\}$$

where G acts on $\mathbb{P}^4$ by the operation $G \times \mathbb{P}^4 \rightarrow \mathbb{P}^4$:

$$(\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4) \cdot (x_0, x_1, x_2, x_3, x_4) = (\zeta_0 x_0, \zeta_1 x_1, \zeta_2 x_2, \zeta_3 x_3, \zeta_4 x_4)$$

Let $\rho : G \times Y_\psi \rightarrow Y_\psi$ be the restriction of the action to Y_ψ , where $y_1 \sim y_2$ if $\rho(g, y_1) = y_2$ for some $g \in G$. Show that the relation Y_ψ defined by the action of G is an equivalence relation.

Proof. For reflexivity, note that

$$\begin{aligned} & (\zeta_0 x_0)^5 + (\zeta_1 x_1)^5 + (\zeta_2 x_2)^5 + (\zeta_3 x_3)^5 + (\zeta_4 x_4)^5 - 5\psi \zeta_0 x_0 \zeta_1 x_1 \zeta_2 x_2 \zeta_3 x_3 \zeta_4 x_4 = \\ & (\zeta_0 x_0)^5 + (\zeta_1 x_1)^5 + (\zeta_2 x_2)^5 + (\zeta_3 x_3)^5 + (\zeta_4 x_4)^5 - 5\zeta_0 \zeta_1 \zeta_2 \zeta_3 \zeta_4 \psi x_0 x_1 x_2 x_3 x_4 = \\ & x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - \left(\prod_{i=0}^4 \zeta_i \right) 5\psi x_0 x_1 x_2 x_3 x_4 = \\ & x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4 \end{aligned}$$

so $y \sim y$ for any $y \in Y_\psi$.

For symmetry, if $y_1 \sim y_2$, then there exists some 5-tuple $g := (\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4)$ such that $\rho(g, y_1) = y_2$. We know none of the ζ_i can be zero, since $\zeta_i^5 = 1$; thus we have $g^{-1} = (\zeta_0^{-1}, \zeta_1^{-1}, \zeta_2^{-1}, \zeta_3^{-1}, \zeta_4^{-1})$ where $\rho(g^{-1}, y_2) = y_1$, so $y_2 \sim y_1$.

For transitivity, if $y_1 \sim y_2$ and $y_2 \sim y_3$, then there exist g_1, g_2 such that $\rho(g_1, y_1) = y_2$ and $\rho(g_2, y_2) = y_3$. Then $g_1 g_2$ fulfills $\rho(g_1 g_2, y_1) = y_3$, so $y_1 \sim y_3$. $\square$

Question 13.5. Let $F(x_0, \dots, x_4)$ be a homogeneous degree 5 polynomial whose zero set in $\mathbb{P}^4$ is X , and we can find local complex coordinates $(z_1, \dots, z_4)$ such that X is locally defined by $f(z_1, z_2, z_3, z_4) = 0$, where f is a holomorphic function. We define the residue, assuming loss of generality that $\frac{\partial f(z)}{\partial z_1} \neq 0$ by the Implicit Function Theorem, as

$$\Omega = \frac{g(z_1, z_2, z_3, z_4) dz_2 \wedge dz_3 \wedge dz_4}{\partial f(z)/\partial z_1} \Big|_X$$

(NOTE: the text has F in the numerator of Ω instead of g , but I'm pretty sure this is wrong.)

a Show that the definition of the residue is independent of the choice of coordinates.

b Show that the holomorphic 3-form Ω has no zeros.

Proof. a In different coordinates $(z'_1, \dots, z'_4)$ on x where f defines X , we suppose X is defined by $f(z'_1, \dots, z'_4) = 0$ for a holomorphic f , where we assume without loss of generality that $\frac{\partial f(z')}{\partial z'_1}$. Then

$$\Omega' = \frac{g(z_1, \dots, z_4) dz'_2 \wedge dz'_3 \wedge dz'_4}{\partial f'(z')/\partial z'_1} \Big|_X$$

Notice that

$$\begin{aligned} d_z f \wedge \Omega &= g(z_1, \dots, z_4) dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \\ d_{z'} f \wedge \Omega' &= g(z_1, \dots, z_4) dz'_1 \wedge dz'_2 \wedge dz'_3 \wedge dz'_4 \\ &= g(z_1, \dots, z_4) J dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \\ &= J d_z f \wedge \Omega' \end{aligned}$$

Thus both sides acquire a Jacobian determinant factor of the change $\{z_i\} \rightarrow \{z'_i\}$, leaving Ω coordinate-invariant:

$$Jd_z f \wedge \Omega' = Jd_z f \wedge \Omega$$

- b From the Implicit Function Theorem, we know that $df \neq 0$. Furthermore, the text gives

$$\Lambda := x_0 dx_1 \wedge \dots \wedge dx_4 + \dots + x_4 dx_0 \wedge \dots \wedge dx_3,$$

where Λ is then expressed locally as

$$\frac{g(z_1, \dots, z_4) dz_1 \wedge \dots \wedge dz_4}{f(z)}$$

near X .

Since the globally-defined Λ is nonvanishing on $\mathbb{P}^4$, $g(z_1, \dots, z_4)$ must be nonvanishing on the neighborhood of X for which Λ is expressed in g . Therefore g is nonvanishing, and therefore $df \wedge \Omega$ is nonvanishing. Thus Ω must be nonvanishing. □

Question 13.6. Prove that S^1 is isomorphic to $\overline{S^1}$ as oriented manifolds.

Proof. An isomorphism in the category of manifolds is a diffeomorphism, so we seek to prove that S^1 is diffeomorphic to $\overline{S^1}$.

Define S^1 in coordinates $t \in \mathbb{R}$ by $(\cos(t), \sin(t)) \in \mathbb{R}^2$. The tangent bundle is then $(-\sin(t), \cos(t))$. If we define another manifold in these coordinates by $(-\cos(t), -\sin(t)) \in \mathbb{R}^2$, then the tangent bundle is $(\sin(t), -\cos(t))$, which is the tangent bundle of our S^1 but with opposite sign. Thus it is $\overline{S^1}$.

The map from S^1 to $\overline{S^1}$ in these definitions is given by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ with its inverse being itself. This is a bijection, and the derivative is itself. Thus S^1 and $\overline{S^1}$ are diffeomorphic. □

Question 13.7. Find a natural one-to-one correspondence between $(V^*)^{\otimes r} \otimes V^{\otimes s}$ and the set of linear transformations $V^{\otimes r} \rightarrow V^{\otimes s}$.

Proof. Let $w_i(\cdot)$ be an element of V^* and v^i be an element of V . An element of $(V^*)^{\otimes r} \otimes V^{\otimes s}$ is given by

$$w_1(\cdot) \otimes \dots \otimes w_r(\cdot) \otimes v^1 \otimes \dots \otimes v^s$$

For any element $u^1 \otimes \dots \otimes u^r \in V^{\otimes r}$, we have

$$\begin{aligned} u^1 \otimes \dots \otimes u^r &\rightarrow w_1(u^1) \otimes \dots \otimes w_r(u^r) \otimes v^1 \otimes \dots \otimes v^s \\ &= 1 \otimes \dots \otimes 1 \otimes \prod_{i=1}^r w_i(u^i) v^1 \otimes \dots \otimes v^s \\ &\rightarrow \prod_{i=1}^r w_i(u^i) v^1 \otimes \dots \otimes v^s \in V^{\otimes s} \end{aligned}$$

Since the tensor product is linear, we have that $w_1(\cdot) \otimes \dots \otimes w_r(\cdot) \otimes v^1 \otimes \dots \otimes v^s$ linearly maps elements from $V^{\otimes r}$ to $V^{\otimes s}$. $\square$

Question 13.8. Use the definition

$$\langle \omega_1, \omega_2, \omega_3 \rangle_\beta := \#\{f : \mathbb{P}^1 \rightarrow X \mid f_*[\mathbb{P}^1] = \beta, f(\mathbb{P}^1) \cap Z_i \neq \emptyset, i = 1, 2, 3\}$$

for $Z_i \subset X$ complex submanifolds of X representing cohomology classes $\omega_i \in H^*(X)$ and $\beta \in H_2(X, \mathbb{Z})$, and plane geometry to show

$$\langle H^2, H^2, H \rangle_1 = 1$$

on $\mathbb{P}^2$, where $H \in H^2(\mathbb{P}^2)$ is the hyperplane class.

Proof. We know from Chapter 6 that $H^*(\mathbb{P}^2) = \mathbb{Z}[H]/(H^3)$. Furthermore, we have the cellular decomposition $\mathbb{P}^0 \subset \mathbb{P}^1 \subset \mathbb{P}^2$. Computing $H^2 \in H^4(\mathbb{P}^2)$, we take two transversely intersecting hyperplanes and intersect, which results in $\mathbb{P}^0$, the generating cell of $H^4(\mathbb{P}^2)$, i.e. $1 \in H^4(\mathbb{P}^2)$. For $f_*[\mathbb{P}^1] = 1$, we must have the image $f(\mathbb{P}^1)$ be the generator $[\mathbb{P}^1]$ of $H_2(\mathbb{P}^2, \mathbb{Z})$. Thus we are looking for how many $\mathbb{P}^1$ images intersect with two $\mathbb{P}^0$ s (points) and one hyperplane.

We will set H as the zero locus of $a_0x_0 + a_1x_1 + a_2x_2$, and the first H^2 as $x_0 = b$ and the second H^2 as $x_1 = c$. There is only $(x_0, x_1) = (b, c)$ that satisfies these conditions, as then $x_2 = \frac{-a_0b - a_1c}{a_2}$. Thus there is only one map, and $\langle H^2, H^2, H \rangle_1 = 1$. $\square$

14 Quantum Cohomology and Enumerative Geometry

Question 14.1. *Use the splitting principle to prove that for a rank r holomorphic vector bundle E we have $c_1(\wedge^r E) = c_1(E)$.*

Proof. Choose a basis $\{e_1, \dots, e_r\}$ of E such that $c_1(e_i) = \alpha_i$. The splitting principle states that $c(E) = \prod_{i=1}^r (1 + \alpha_i)$ for $\alpha_i \in H^2(X)$, for X the manifold E is over. Taking the single-power terms of the α_i for c_1 , we get $c_1(E) = \sum_i \alpha_i$.

Note that the dimension of $\wedge^r E$ is $\binom{r}{r} = 1$, and give the basis as $e_1 \wedge \dots \wedge e_r$. We map this to the element $e_1 \otimes \dots \otimes e_r$. By the splitting principle, we can think of this as an element of line bundles $L_1 \otimes \dots \otimes L_r$, giving us $c(\wedge^r E) = (1 + \alpha_1 + \dots + \alpha_r)$. Taking the single-power terms to get c_1 , we have $c_1(\wedge^r E) = \sum_i \alpha_i = c_1(E)$. $\square$

Question 14.2. *Check*

$$\dim \overline{M}_{0,n}(\mathbb{P}^N, d) = Nd + N + d + n - 3$$

for the dimension of $\overline{M}_{0,n}(\mathbb{P}^N, d)$.

Proof. The space $\overline{M}_{0,n}(\mathbb{P}^N, d)$ is the set of isomorphism classes of maps $f : (C, p_1, \dots, p_n) \rightarrow \mathbb{P}^N$ with C a tree of rational curves satisfying the the properties:

1. $f_*[C] = d[\text{line}]$
2. the $p_i \in C$ are distinct from each other and nodes of C
3. If $f_i = f|_{C_i}$ is constant, then the component $C_i \in C$ is required to have at least three special points (nodes or p_i).

To consider the space of isomorphism classes, we must subtract the dimension of the space of automorphisms $\mathbb{P}^1 \rightarrow \mathbb{P}^1$. These automorphisms are of the form

$$(x_0, x_1) \mapsto (a_0x_0 + a_1x_1, b_0x_0 + b_1x_2)$$

i.e. are indexed by (a_0, a_1, b_0, b_1) . Automorphisms without special points are therefore in one-to-one correspondence with points in $\mathbb{P}^3$, a 3-dimensional space.

The dimension of the space of degree d maps from a single component $C \cong \mathbb{P}^1$ to $\mathbb{P}^N$ is found the same way: A degree d map from $\mathbb{P}^1$ to $\mathbb{P}^N$ has $N + 1$

maps, one for each coordinate of $\mathbb{P}^N$. These maps are homogeneous degree d polynomials in two variables, which are indexed by $d + 1$ coefficients each. Thus a map from $\mathbb{P}^1$ to $\mathbb{P}^N$ is indexed by $(N + 1)(d + 1)$ coefficients, which are in one-to-one correspondence with $\mathbb{P}^{Nd+d+N+1-1}$. So the dimension of the space of isomorphism classes of maps $\mathbb{P}^1 \rightarrow \mathbb{P}^N$ with n marked points is given by

$$Nd + d + N - 3 + n$$

This is the dimension in the case where C is a single component.

Suppose instead that C has a node, i.e. has two components, with the mapping to $\mathbb{P}^N$ having degrees d_1, d_2 , respectively, with $d_1 + d_2 = d$. Then, summing the contributions of both components, gives $Nd + d + 2N - 2 * (3) + n + 1$, where the extra $+1$ dimension comes from counting the node in common between the two components. Imposing the condition that the image of the node in $\mathbb{P}^N$ matches under both images of the components introduces one further dimension constraint per coordinate:

$$\begin{aligned} & a_0 y_0^{d_1} + a_1 y_0^{d_1-1} y_1 + \dots + a_{d_1} y_1^{d_1}, \\ & b_0 y_0^{d_2} + b_1 y_0^{d_2-1} y_1 + \dots + b_{d_2} y_1^{d_2} = \text{anything} \\ \Rightarrow & a_0 y_0^{d_1} + a_1 y_0^{d_1-1} y_1 + \dots + a_{d_1} y_1^{d_1} = b_0 y_0^{d_2} + b_1 y_0^{d_2-1} y_1 + \dots + b_{d_2} y_1^{d_2} \end{aligned}$$

Imposing this constraint thus introduces N dimension reductions, and the dimension of the moduli space for a two-component tree is $Nd + d + N - 5 + n$. This reduces the dimension from that of the case where C is single-component. Thus cases of trees with nonzero nodes are lower-dimensional subspaces of an $(Nd + d + N - 3 + n)$ -dimensional space.

Therefore, $\dim \overline{M}_{0,n}(\mathbb{P}^N, d) = Nd + N + d + n - 3$. $\square$

Question 14.3. Show that the virtual dimension formula

$$D = \langle c_1(TX), \beta \rangle + \dim X - 3 + n$$

follows from the dimension formula $\dim \overline{M}_{0,n}(\mathbb{P}^N, d) = Nd + N + d + n - 3$ for $\mathbb{P}^N$.

Proof. Since the dimension formula gives the dimension of the space of degree d maps $f : \mathbb{P}^1 \rightarrow \mathbb{P}^N$ minus the dimension of automorphisms, plus the dimension for each given point. It remains to find the dimension of the space of degree d

maps $f : \mathbb{P}^1 \rightarrow \mathbb{P}^N$. This is asserted without proof in the book as $\langle c_1(T\mathbb{P}^N), \beta \rangle$, so I assume this problem is asking to prove that this is the dimension we seek.

In $\mathbb{P}^N$, β is given by $d[\text{line}]$, where $[\text{line}]$ is the homology class of a 1-dimensional subspace of $\mathbb{P}^N$, i.e. hyperplane. If $H \in H^2(\mathbb{P}^N)$ is the cohomology class of a generic hyperplane, we have $\int_{[\text{line}]} H = 1$, since the integral counts the intersection points of a generic hyperplane with another generic hyperplane.

Recall from Chapter 8 that $c_1(T\mathbb{P}^N) = (N + 1)H$, so $\langle c_1(TX), \beta \rangle = (N + 1)d \int_{[\text{line}]} H = (N + 1)d$, i.e. a degree d curve intersects the $(N + 1)$ twists of the hyperplane class $(N + 1)d$ times.

If $f : \mathbb{P}^1 \rightarrow \mathbb{P}^N$ defines an irreducible rational curve C of degree d such that $\beta = [C]$, then

$$\langle c_1(T\mathbb{P}^N), \beta \rangle = \int_C c_1(T\mathbb{P}^N) = \int_{\mathbb{P}^1} f^* c_1(T\mathbb{P}^N) = \langle f^* c_1(T\mathbb{P}^N), [\mathbb{P}^1] \rangle$$

where $[\mathbb{P}^1] \in H_2(\mathbb{P}^1)$ is $\mathbb{P}^1$'s fundamental class, and the second equality is the pullback asserted in Chapter 12. This gives the tangent space along which f can vary, which is $(N + 1)d$.

However, we are currently undercounting, because we are integrating over the homology class $[\mathbb{P}^1]$, which encompasses more maps ruled out by homology's deformation-invariance. Thus we need to add the number of dimensions which are covered by $[\mathbb{P}^1]$.

Since $[\mathbb{P}^1]$ corresponds to the class of any point $[x] \in H^0(\mathbb{P}^1)$ via Poincaré Duality, we then examine the dimension of where a point x is allowed to go in the image of f . Since this gets mapped to $\mathbb{P}^N$, $[x]$ can range over all of $\mathbb{P}^N$, so we add $\dim \mathbb{P}^N = N$ to our number of dimensions of the space of our f .

Thus $\langle c_1(T\mathbb{P}^N), \beta \rangle + \dim \mathbb{P}^N$ is the expected dimension of the space of maps $f : \mathbb{P}^1 \rightarrow \mathbb{P}^N$. (This is the best justification I can give using only the content of the book.) $\square$

Question 14.4. *Compute all of the quantum products of pairs of elements from the basis $(1, H, H^2)$ for the cohomology of $\mathbb{P}^2$. Verify that the quantum product is associative.*

Proof. It's given in the text that the cohomology of $\mathbb{P}^2$ is spanned by the basis $(T_0, T_1, T_2) = (1, H, H^2)$, with dual basis $(T^0, T^1, T^2) = (H^2, H, 1)$ respectively, and intersection product given by $\langle H^a, H^b \rangle = \delta_{a,2-b}$. 1 is the cohomology class of $\mathbb{P}^2$ itself, H is the cohomology class of a hyperplane, and H^2 is the cohomology

class of a point. We also have

$$T_i * T_j = \sum_r \langle T_i, T_j, T_r \rangle T^r$$

By the dimension axiom, $\langle H^{a_1}, H^{a_2}, H^{a_3} \rangle$ is zero unless $a_1 + a_2 + a_3 = 3d + 2$, where d is the degree of the image of the curve.

For $1 * 1, 1 * H, 1 * H^2, H * 1, H^2 * 1, H * H$, by the dimension axiom, the only three-point functions that are not zero have two H^2 arguments and one H argument. Therefore, $d = 0$ for all of pairings, and we have $\langle T_i, T_j, T_r \rangle = \int_{\overline{M}_{0,3}(\mathbb{P}^2, 0)} [ev_1^*(T_i) ev_2^*(T_j) ev_3^*(T_r)]$. $\overline{M}_{0,3}(\mathbb{P}^2, 0)$ is just the space of constant maps from (C, p_1, p_2, p_3) to $\mathbb{P}^2$, which is classified by the image point in $\mathbb{P}^2$, i.e. $\mathbb{P}^2$ itself. The integrand, as we know from the text, in this case is just

$$[\{f : (C, p_1, p_2, p_3) \rightarrow \mathbb{P}^2 | f(p_i) \in T_i\}]$$

Since the f s are constant, all p_i must map to a single point, so this cohomology element is simply given by $[T_i \cap T_j \cap T_r]$. Thus the quantum product for these pairings is given by

$$\int_{\mathbb{P}^2} T_i \cdot T_j \cdot T_r$$

We now compute the quantum products:

- $1 * 1$: As argued above, this is given by $(\int_{\mathbb{P}^2} 1 \cdot 1 \cdot 1) H^2 + (\int_{\mathbb{P}^2} 1 \cdot 1 \cdot H) H + (\int_{\mathbb{P}^2} 1 \cdot 1 \cdot H^2) 1$, which, when written in terms of cohomology pairing, is $\langle 1, 1 \rangle H^2 + \langle H, 1 \rangle H + \langle H^2, 1 \rangle 1$. Applying the pairing rule for $\mathbb{P}^2$ above, only the $\langle H^2, 1 \rangle 1 = 1$ term remains. Thus $1 * 1 = 1$.
- $1 * H$: In the same way as above, this pairing is given by

$$\begin{aligned} \left(\int_{\mathbb{P}^2} 1 \cdot H \cdot 1 \right) H^2 + \left(\int_{\mathbb{P}^2} 1 \cdot H \cdot H \right) H + \left(\int_{\mathbb{P}^2} 1 \cdot H \cdot H^2 \right) 1 \\ = \langle H, 1 \rangle H^2 + \langle H^2, 1 \rangle H + \langle 1, 1 \rangle 1 \\ = H \end{aligned}$$

Thus $1 * H = H$.

- $1 * H^2$:

$$\begin{aligned} \left(\int_{\mathbb{P}^2} 1 \cdot H^2 \cdot 1 \right) H^2 + \left(\int_{\mathbb{P}^2} 1 \cdot H^2 \cdot H \right) H + \left(\int_{\mathbb{P}^2} 1 \cdot H^2 \cdot H^2 \right) 1 \\ = \langle H^2, 1 \rangle H^2 + \langle 1, 1 \rangle H + \langle H, 1 \rangle 1 \\ = H^2 \end{aligned}$$

Thus $1 * H^2 = H^2$.

- $H * 1, H^2 * 1$: Note that, due to commutativity of the intersection pairing, the quantum product, at least in these cases, is commutative as well, so $H * 1 = H$ and $H^2 * 1 = H^2$.

- $H * H$:

$$\begin{aligned} \left(\int_{\mathbb{P}^2} H \cdot H \cdot 1 \right) H^2 + \left(\int_{\mathbb{P}^2} H \cdot H \cdot H \right) H + \left(\int_{\mathbb{P}^2} H \cdot H \cdot H^2 \right) 1 \\ = \langle H^2, 1 \rangle H^2 + \langle 1, 1 \rangle H + \langle H, 1 \rangle 1 \\ = H^2 \end{aligned}$$

Thus $H * H = H^2$.

- $H^2 * H$: by the dimension axiom, if $d = 1$ there can be nonzero terms in the sum, so the quantum product is given by

$$\begin{aligned} \langle H^2, H, 1 \rangle_0 H^2 + \langle H^2, H, H \rangle_0 H + \langle H^2, H, H^2 \rangle_1 q 1 = \\ \langle H^3, 1 \rangle H^2 + \langle H^4, 1 \rangle H + \langle H^2, H, H^2 \rangle_1 q 1 = \\ \langle 1, 1 \rangle H^2 + \langle H, 1 \rangle H + \langle H^2, H, H^2 \rangle_1 q 1 = \\ \langle H^2, H, H^2 \rangle_1 q 1 \end{aligned}$$

From problem 8 in the previous chapter, we know that $\langle H^2, H, H^2 \rangle_1 = 1$, since intersection is commutative. Thus $H^2 * H = q$.

- $H * H^2$: Due to commutativity in the zero-degree correlation functions and intersection in the one-degree correlation function, $H * H^2 = q$.
- $H^2 * H^2$ is computed in the book as qH .

Thus our multiplication table is

	1	H	H^2
1	1	H	H^2
H	H	H^2	q
H^2	H^2	q	qH

For associativity, we note that the group cohomology without q factors is just regular group multiplication, which is associative by axiom. Furthermore, we have

$$\begin{aligned}
H^2 * (H * H) &= H^2 * H^2 = qH \\
(H^2 * H) * H &= q * H = qH, \\
H^2 * (H^2 * H) &= H^2 * q = qH^2 \\
(H^2 * H^2) * H &= qH * H = qH^2
\end{aligned}$$

Because the pairing, as established before, is abelian, the quantum pairing is associative. $\square$

Question 14.5. *Verify the Gromov-Witten potential for $\mathbb{P}^1$.*

Proof. Pick the basis $(T_0, T_1) = (1, H)$ of the cohomology $H^*(\mathbb{P}^1)$, with H denoting the hyperplane class (the cohomology class of a point). From the previous problem, $\langle H, 1, 1 \rangle = \int_{\mathbb{P}^1} H = 1$ is the only nonzero degree-zero Gromov-Witten invariant.

For higher-degree Gromov-Witten invariants, the dimension axiom gives $D = 2d + n - 2$. By the dimension axiom, for an n -point correlation function, we must have $\sum_i n_i = 2d - 2 + n$. Since all $n_i < 2$, the only degree that works is $d = 1$ and each cohomology class must be H .

Since each H represents the cohomology of a point, we'll label each of the n H s by distinct points q_i . The only degree 1 stable map $f : (C, p_1, \dots, p_n) \rightarrow \mathbb{P}^1$ with $f(p_i) = q_i$ for each i is the isomorphism $C \rightarrow \mathbb{P}^1$, and we must set p_i to $f^{-1}(q_i)$. This is the only stable map. It's given in the text that

$$ev_1^*(\omega_1) \dots ev_n^*(\omega_n) = [\{f : (C, p_1, \dots, p_n) \rightarrow X \mid f(p_i) \in H, i = 1, \dots, n\}]$$

Since, H is the cohomology class of a point in $\mathbb{P}^1$, this is just the class of maps with $f(p_i) = q_i$ for all i , which is our single stable map. Thus $\langle H, \dots, H \rangle_1 = 1$.

Therefore, for an element $\gamma = t_0 + Ht_1 \in H^*(\mathbb{P}^1)$, we have $\langle t_0 + t_1 H, t_0 + t_1 H, t_0 + t_1 H \rangle_0 = 6t_1 t_0^2 \langle H, 1, 1 \rangle_0 = 3t_1 t_0^2$ and n -point correlation functions $\langle t_0 + t_1 H, \dots, t_0 + t_1 H \rangle_1 = t_1^n \langle H, \dots, H \rangle_1 = t_1^n$ by linearity of integrals.

$$\begin{aligned}
\Phi(\gamma) &= \frac{1}{n!} \sum_{n \geq 3} \sum_{\beta} \langle \gamma, \dots, \gamma \rangle_{\beta} q^{\beta} \\
&= \frac{1}{2} t_1 t_0^2 + \sum_{n > 3} \frac{t_1^n}{n!} q
\end{aligned}$$

□

Question 14.6. *Verify the form of the Gromov-Witten potential for $\mathbb{P}^2$.*

Proof. With $(1, H, H^2)$ our cohomology basis of $\mathbb{P}^2$, we first compute the zero-degree Gromov-Witten invariants. As we computed a problem ago, we have

$$\langle H^2, 1, 1 \rangle_0 = \int_{\mathbb{P}^2} H^2 \cdot 1 \cdot 1$$

which is just the integral of a point over all of $\mathbb{P}^2$, which gives 1. Similarly, the $\langle H, H, 1 \rangle_0$ gives the integral of the intersection of two hyperplanes, i.e. a point of $\mathbb{P}^2$, over $\mathbb{P}^2$, which again is 1. Thus the powers of H in the three-point correlation functions must add to 2 mod 3 to give the cohomology class of a point, and thus a nonzero integral. These, up to permutation, are the only correlation functions that do this.

For nonzero d , the virtual dimension of $\overline{M}_{0,n}(\mathbb{P}^2, d)$ is $3d - 1 + n$. We seek to find

$$\langle \omega_1, \dots, \omega_n \rangle_d = \int_{\overline{M}_{0,n}(\mathbb{P}^2, d)} ev_1^*(\omega_1) \dots ev_n^*(\omega_n)$$

If any of the ω_i are equal to $1 = [\mathbb{P}^2]$, then the i^{th} point can go anywhere in $\mathbb{P}^2$. Thus our moduli space becomes $\overline{M}_{0,n-1}(\mathbb{P}^2, d)$, and decreases a dimension. However, $ev_i^*(1) = [ev_i^{-1}(\mathbb{P}^2)]$. Since the i^{th} marked point can range over all of $\mathbb{P}^2$, the inverse of ev_i is the entirety of $\overline{M}_{0,n}(\mathbb{P}^2, d)$. Thus $ev_i^*(1) = [\overline{M}_{0,n}(\mathbb{P}^2, d)]$, so the integrand doesn't reduce the dimension. Thus the integral is 0 if one of the $\omega_i = 1$.

As detailed in the text, define $K_d = \langle H^2, \dots, H^2 \rangle_d$ as the number of rational curves in $\mathbb{P}^2$ of degree d passing through $3d - 1$ general points. The virtual dimension of $\overline{M}_{0,n}(\mathbb{P}^2, d)$ is $3d + n - 1$. If there are c copies of H^2 in our n -point correlation function, then there are $n - c$ copies of H as well. Then the dimension

axiom gives $2c + n - c = 3d + n - 1$, so there must be $c = 3d - 1$ copies of H^2 in our n -point correlation function:

$$\underbrace{\langle H^2, \dots, H^2, H, \dots, H \rangle_d}_{3d-1}$$

Each of the H^2 are represented by $q_1, \dots, q_{3d-1}$ points, distinct because they are in general position. Each of the $n - 3d + 1$ copies of H are lines L_i in general position. For a degree d stable map $f : (C, p_1, \dots, p_n) \rightarrow \mathbb{P}^2$, the first $3d - 1$ marked points must map to the q_i . Then f must map C to one of the K_d rational curves of degree d which pass through the points q_i . Proceeding as in the text, fix one of these K_d rational curves and call it D . The last $n - 3d + 1$ marked points must map to their respective lines L_i , but they must also be contained in D . By Bézout's Theorem, each $L_i \cap D$ consists of d points. If we choose the corresponding p_i to be the preimage of one of these d points under f , we have

$$\langle H^2, \dots, H^2, H, \dots, H \rangle_d = d^{n-3d+1} K_d$$

Putting all of these correlation functions together, we have, for $\gamma = t_0 + t_1 H + t_2 H^2 \in H^*(\mathbb{P}^2)$,

$$\begin{aligned} \langle \gamma, \gamma, \gamma \rangle_0 &= t_2 t_0^2 (\langle H^2, 1, 1 \rangle_0 + \langle 1, H^2, 1 \rangle_0 + \langle 1, 1, H^2 \rangle_0) \\ &\quad + t_1^2 t_0 (\langle H, H, 1 \rangle_0 + \langle H, 1, H \rangle_0 + \langle 1, H, H \rangle_0) \\ &= 3 (t_2 t_0^2 + t_1^2 t_0), \\ \underbrace{\langle \gamma, \dots, \gamma \rangle_d}_n &= t_1^{n-3d+1} t_2^{3d-1} \binom{n}{3d-1} \underbrace{\langle H^2, \dots, H^2, H, \dots, H \rangle_d}_{3d-1} \\ &= t_1^{n-3d+1} t_2^{3d-1} \binom{n}{3d-1} d^{n-3d+1} K_d \end{aligned}$$

since there are $\binom{n}{3d-1}$ ways to arrange $3d - 1$ copies of H^2 in an n -point correlation function. Plugging this into our Gromov-Witten potential, we get

$$\begin{aligned} \Phi(\gamma) &= \frac{1}{2} t_0^2 t_2 + \frac{1}{2} t_0 t_1^2 + \sum_d t_2^{3d-1} \sum_{n > 3d-1} \frac{1}{n!} (dt_1)^{n-3d+1} \frac{n!}{(3d-1)!(n-3d+1)!} K_d q^d \\ &= \frac{1}{2} t_0^2 t_2 + \frac{1}{2} t_0 t_1^2 + \sum_d t_2^{3d-1} \sum_{n > 3d-1} (dt_1)^{n-3d+1} \frac{1}{(3d-1)!(n-3d+1)!} K_d q^d \end{aligned}$$

Relabeling our second sum $n - 3d + 1 \rightarrow n'$, we have

$$\begin{aligned}
\Phi(\gamma) &= \frac{1}{2}t_0^2t_2 + \frac{1}{2}t_0t_1^2 + \sum_d t_2^{3d-1} \sum_{n>3d-1} \frac{1}{n!} (dt_1)^{n-3d+1} \frac{n!}{(3d-1)!(n-3d+1)!} K_d q^d \\
&= \frac{1}{2}t_0^2t_2 + \frac{1}{2}t_0t_1^2 + \sum_d t_2^{3d-1} \frac{1}{(3d-1)!} K_d q^d \sum_{n'} (dt_1)^{n'} \frac{1}{n'!} \\
&= \frac{1}{2}t_0^2t_2 + \frac{1}{2}t_0t_1^2 + \sum_d K_d \frac{t_2^{3d-1}}{(3d-1)!} e^{dt_1} q^d
\end{aligned}$$

□