Mirror Symmetry

Alec Lau

In Winter Quarter 2020, I did the Directed Reading Program on mirror symmetry, advised by Libby Taylor.

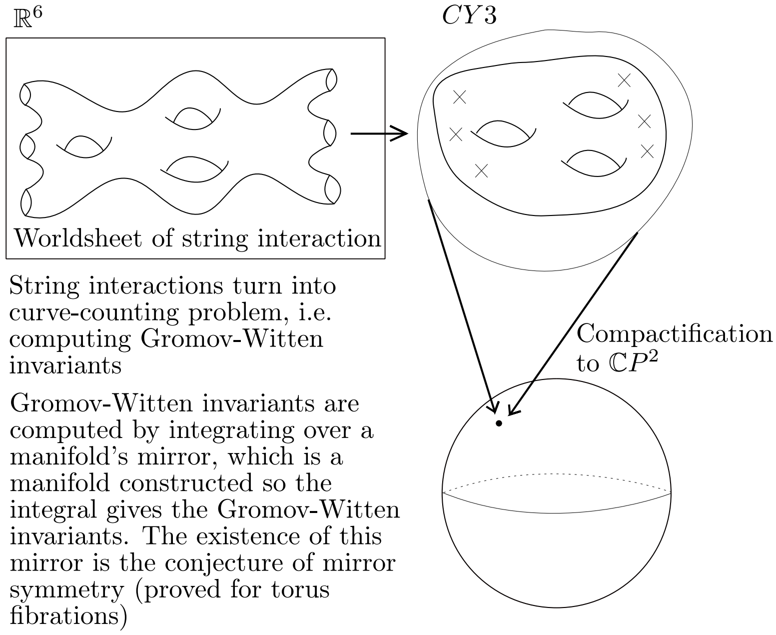

Figure 1: Outline of mirror symmetry

Outline

Mirror symmetry has to do with certain objects called Calabi-Yau manifolds. Calabi-Yau manifolds are kähler manifolds that are Ricci flat.

Physicists like Calabi-Yau manifolds because

- Kähler-ness allows one to have a particular type of invariance of the action on a supersymmetric $\sigma-$model under supersymmetry.
- Ricci flat manifolds map to a point in projective space, and thus provide a nice way to get down from 10 dimensions in string theory to our familiar four dimensions.

Mathematicians like Calabi-Yau manifolds due to mirror symmetry. We examine two types of mirror symmetry: Homological and otherwise.

- The latter states that the Hodge structure of a Calabi-Yau manifold determines the Gromov-Witten invariants of its mirror.
- Homological mirror symmetry states that the derived category of coherent sheaves is equivalent to the Fukaya category on its mirror.

We will unpack these definitions below, in adhering to their order as much as possible.

Contents

1 The Physics

1.1 Quantum Field Theory

We consider fields on a manifold X. A field is an object that can be locally expressed in local coordinates on X. This can be scalars, vectors, sections of vector bundles, maps from X to another manifold Y, etc. If we have a set of fields $\mathcal{F}$, we specify a physical theory by giving an action

$$S : \mathcal{F} \to \mathbb{R} \tag{1}$$

In classical field theory, the equations of motion are derived by minimizing S. In quantum field theory, we integrate over the space of such maps:

$$\int \mathcal{D}f e^{iS(f)},\ f \in \mathcal{F} \tag{2}$$

the $\mathcal{D}f$ measure is typically not well-defined.

Example 1. *Let $X = \mathbb{R}$, for time. The function $x(t)$ on X is the position of a particle of mass m, with an appropriate smoothness condition imposed on all $x(t)$s. For a potential V, we have*

$$S : \mathcal{F} \to \mathbb{R},\ S(x(t)) = \int [\frac{m}{2}(\frac{dx(t)}{dt})^2 - V(x(t))]dt \tag{3}$$

Since there are no "spatial dimensions" and we just have our time dimension, we find that a (0+1)-dimensional quantum field theory is just 1-dimensional quantum mechanics.

1.2 Bosonic and Fermionic Fields

Suppose we have fields X_1, X_2 such that $X_1(p)X_2(q) = X_2(q)X_1(p)$, for p, q points on X. Such fields are called bosonic fields. If instead $X_1(p)X_2(q) = -X_2(q)X_1(p)$, such fields are called fermionic fields. Note that this implies $X_iX_i = 0$ and $(X_iX_j)^k = 0, k \geq 2$. A natural way to think about this is to think of the fields on a manifold as the 1-forms with the multiplication being the wedge product.

Example 2.

$$\psi(x) = \sum_i f_i(x_i)dx_i,\ \phi(x) = \sum_i g_i(x_i)dx_i \tag{4}$$

$$\psi(x)\phi(x) = (\sum_i f_i(x_i)dx_i) \wedge (\sum_i g_i(x_i)dx_i) = -(\sum_i g_i(x_i)dx_i) \wedge (\sum_i f_i(x_i)dx_i) = -\phi(x)\psi(x) \tag{5}$$

Since S determines the physical theory, in order for the theory to be physical, we require that S commutes with all fields. Thus each term in S must have an even number of fermionic fields.

We define fermionic integration as

$$\int (a + b\psi)d\psi = b \tag{6}$$

to deal with such actions.

Example 3. *For a (supersymmetric) action, we let $h(x)$ be a twice-differentiable function of x. Let*

$$S(x, \psi_1, \psi_2) := \frac{h'(x)^2}{2} - h''(x)\psi_1\psi_2 \tag{7}$$

The partition function is then

$$
\begin{aligned}
Z &= \int \exp(-h'(x)^2/2 + h''(x)\psi_1\psi_2)dxd\psi_1 d\psi_2 && (8)\\
(\textit{Taylor Expanding}) \quad &= \int \exp(-h'(x)^2/2)(1 + h''(x)\psi_1\psi_2 + ...)dxd\psi_1 d\psi_2 && (9)\\
((\psi_1\psi_2)^k = 0, k \geq 2) \quad &= \int \exp(-h'(x)^2/2)(1 + h''(x)\psi_1\psi_2)dxd\psi_1 d\psi_2 && (10)\\
(\textit{Fermionic integration}) \quad &= \int h''(x)\exp(-h'(x)^2/2)dx && (11)
\end{aligned}
$$

Thus fermionic fields can be integrated out to have a purely bosonic action.

2 Calabi-Yau Manifolds

2.1 Kähler Manifolds & Hodge Structure

Quantum mechanics works best in complex vector spaces. Why it works out in the complex numbers is worthy of a whole blog post by Prof. Scott Aaronson: `https://www.scottaaronson.com/blog/?p=4021`. Either way, it makes sense for our manifold M to be complex. Since M is complex, we can endow the real tangent bundle $T_M^{\mathbb{R}}$ with an almost-complex (pseudo-complex) structure operator J, satisfying $J^2 = -1$.

Example 4. *J, under the identification $a + ib = (a, b), a, b \in \mathbb{R}^n$, is*

$$\begin{pmatrix} 0 & -Id_n \\ Id_n & 0 \end{pmatrix} \tag{12}$$

An important property of complex $n-$manifolds identified with real $2n-$manifolds is the following decomposition. If $z^k := x^k + iy^k$ are holomorphic coordinates, then $\partial_k = \frac{\partial}{\partial z^k} = \frac{1}{2}(\frac{\partial}{\partial x^k} - i\frac{\partial}{\partial y^k})$ generates a subspace $T'M \subset TM$, and $\partial_{\overline{k}} = \frac{\partial}{\partial \overline{z}^k} = \frac{1}{2}(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k})$ generates a subspace $T''M \subset TM$. With abuse of notation, then, we write $TM = T'M, \overline{T}M = T''M$, so we get $TM \otimes \mathbb{C} = TM \oplus \overline{T}M$ through abuse of notation. We do the same thing with cotangent bundles, so a $(p,q)-$form $\theta \in \Gamma(\Lambda^p T^*M \otimes \Lambda^q \overline{T}^* M)$, so we have the decomposition

$$\Omega^n(M) = \oplus_{p+q=n} \Lambda^p T^*M \otimes \Lambda^q \overline{T}^* M := \oplus_{p+q=n} \Omega^{p,q}(M) \tag{13}$$

Since we've defined differential forms in this way, we want a sensible $d : \Omega^n \to \Omega^{n+1}$ map:

$$d = \partial + \overline{\partial} \tag{14}$$

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M),\ \overline{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}(M) \tag{15}$$

$$\partial\theta = \sum_k \partial_{z^k} \theta_{I,\overline{J}} dz^k \wedge dz^I \overline{d}z^{\overline{J}},\ \overline{\partial}\theta = \sum_k \partial_{\overline{z}^k} \theta_{I,\overline{J}} d\overline{z}^k \wedge dz^I \overline{d}z^{\overline{J}} \tag{16}$$

The condition that $d^2 = 0$ gives us

$$\partial^2 = 0, \overline{\partial}^2 = 0, \partial\overline{\partial} + \overline{\partial}\partial = 0 \tag{17}$$

We now have the tools to construct this version's De Rham cohomology. We define $H^{p,q}_{\overline{\partial}}(M)$ as closed/exact $(p,q)-$forms with respect to $\overline{\partial}$, and $H^{p,q}_{\partial}(M)$ the same way. The Čech-Dolbeault Theorem states that

$$H^{p,q}_{\overline{\partial}}(M) \cong H^q(\Lambda^p T^*M) \tag{18}$$

Note that the condition $\overline{\partial}^2 = 0$ is equivalent to the condition that the Lie bracket ($[X,Y] = (X^a\partial_a Y^b - Y^a \partial_a X^b)\frac{\partial}{\partial x^b}$) of two holomorphic vectors is always a holomorphic vector. This is equivalent to the Nijenhuis tensor ($[JX, JY] - J[X, JY] - J[JX, Y] - [X, Y]$) vanishing.

A Hermitian metric is a positive-definite inner product $TM \otimes \overline{T}M \to \mathbb{C}$ at every point of M. Call this hermitian metric h. We also want this metric to be compatible with our identification with J. Thus we have the two conditions

$$h(u, Jv) = -ih(u,v), h(u,v) = \overline{h(v,u)} \tag{19}$$

where the last condition is due to h being hermitian (think $\langle x|y\rangle = \langle y|x\rangle^*$). With the analogue of $a + bi \in M$, we can decompose h into $g - i\omega$, with g, ω real and $J-$invariant. Thus the two conditions become

$$\omega(u, Jv) = g(u, v), \omega(u, v) = -\omega(v, u) \tag{20}$$

If ω is closed, h is called a Kähler metric, and M is a Kähler manifold. With this nondegenerate 2-form, we can define $\partial^\dagger, \overline{\partial}^\dagger$ using h:

$$\partial^\dagger : \Omega^{p,q} \to \Omega^{p-1,q},\ \overline{\partial}^\dagger : \Omega^{p,q} \to \Omega^{p,q-1} \tag{21}$$

$$(\theta, \partial\psi) = (\partial^\dagger\theta, \psi),\ (\theta, \overline{\partial}\psi) = (\overline{\partial}^\dagger\theta, \psi) \tag{22}$$

$$\Delta_\partial = \partial\partial^\dagger + \partial^\dagger\partial,\ \Delta_{\overline{\partial}}\overline{\partial\partial}^\dagger + \overline{\partial}^\dagger\overline{\partial} \tag{23}$$

For a Kähler metric, we have

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\overline{\Delta}} \tag{24}$$

Thus the De Rham cohomology decomposes into $\overline{\partial}$ cohomology, so we have the **Hodge Decomposition**:

$$H^r(M) = \oplus_{p+q=r} H^{p,q}(M) = \oplus_{p+q=r} H^q(M; \Omega^p) \tag{25}$$

(the data on the right is called the **Hodge Structure**)

2.2 Ricci Flatness

Suppose M is an $n-$manifold. Mapping M to projective space $\mathbb{P}^n$ is given by the top form on M: ω. However, $\omega = \wedge^n\Omega$, where Ω is a line bundle. A line bundle gives sections on $M : (f_0, ..., f_N)$. When mapping to $\mathbb{P}^n$, we map sections to points:

$$(f_0, ..., f_N) \mapsto [x_0, ..., x_N] \tag{26}$$

However, when a manifold is Ricci flat, there is only one section. Therefore we have

$$1 \mapsto [1, 0, ..., 0] \tag{27}$$

This is obviously invariant under isomorphism.

3 Mirrors

4 String Theory

Consider a smooth compact surface (2-manifold) Σ. The reason that this is called a "string theory" is because since Σ is closed, it can be sliced into collections of circles, i.e. "strings." Σ is the worldsheet of these circles.

4.1 Bosonic String Action

Bosonic fields are the C^∞ maps

$$f : \Sigma \to \mathbb{R}^n \tag{28}$$

and a Riemannian metric on Σ. In defining this theory, we define the action as

$$S(f, g) = -\frac{1}{2\pi\alpha'} \int_\Sigma \sum_{i=1}^{n} \sum_{j,k=1}^{2} (g^{jk}(x) \frac{\partial f_i}{\partial x_j} \frac{\partial f_i}{\partial x_k})(\sqrt{g(x)}) dx_1 \wedge dx_2 \tag{29}$$

where $(\sqrt{g(x)})dx_1 \wedge dx_2$ is the area form, α' is the square of the string scale l_s, and g^{jk} is the j, k^{th} entry of the inverse of the metric g_{jk}.

For maps $f = \sum_i \phi_i : \Sigma \to \mathbb{C}^n$, we have the action

$$S(f, g) = -\frac{i}{2\pi\alpha'} \int \sum_{j=1}^{n} (\frac{\partial \phi_j}{\partial z} \frac{\partial \overline{\phi_j}}{\partial \overline{z}} + \frac{\partial \overline{\phi_j}}{\partial z} \frac{\partial \phi_j}{\partial \overline{z}}) dz \wedge d\overline{z} \tag{30}$$

where, as above, $\frac{\partial}{\partial z_j} := \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$, $\frac{\partial}{\partial \overline{z_j}} := \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$. (Thus $\{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z_i}}\}$ for $T_x\mathbb{C}^n \otimes \mathbb{C}, x \in \mathbb{C}^n$).

More generally, a bosonic string action can also be formulated for maps from Σ to an arbitrary manifold with a Riemannian metric.

4.2 The A-Model

For a supersymmetric string theory (superstring theory) we need a string theory for maps from Σ to a Kähler manifold X. The A-model has a fermionic symmetry analogous to the supersymmetry of the superstring. The fermions are

- $\chi(z)$, a C^∞ section of $f^*TX \otimes \mathbb{C}$: $\chi = \sum_{I=1}^{2n} \chi^I(z) f^*(\frac{\partial}{\partial x_I})$
- $\psi_z(z)$, a C^∞ section of $(T^{1,0}\sigma)^* \otimes f^*T^{0,1}X$: $\psi_z = \sum_{i=1}^{n} \psi_z^{\bar{i}}(z) dz \otimes f^*(\frac{\partial}{\partial \bar{z}_i})$
- $\psi_{\bar{z}}(z)$, a C^∞ of $(T^{0,1}\Sigma)^* \otimes f^*T^{1,0}X$: $\psi_{\bar{z}}^i = \sum_{i=1}^{n} \psi_{\bar{z}}^i(z) d\bar{z} \otimes f^*(\frac{\partial}{\partial z_i})$

The A-model action is

$$\frac{1}{2\pi\alpha'} \int_\Sigma (\sum_{i,j=1}^{n} h_{i\bar{j}}(f(z))(\psi_z^{\bar{j}} \frac{\partial \chi^i}{\partial \bar{z}} + \psi_{\bar{z}}^i \frac{\partial \chi^{\bar{j}}}{\partial z})) dz \wedge d\bar{z} \tag{31}$$

where h_{ij} is Kähler.

5 Gromov-Witten Invariants

5.1 Why projective space?

Algebraic geometers want to work in a space that is nice when it comes to root counting of polynomials. The most obvious thing to do, then, is work in the complex field, which is algebraically closed. Now we want a place where, for instance, two lines will always intersect at a point. In general they do, but if they are parallel, we'll need to add a point at infinity. If they intersect infinitely, they'll be the same line, so we quotient by field multiplication. The resulting space is $\mathbb{C}^n$. This is a manifold, given by charts $(x_0, ..., x_n) \to [\frac{x_0}{x_i}, ..., \frac{x_0}{x_i}]$ in the domain where $x_i \neq 0$.

Remark 1. *The point at infinity exists because, for instance, if we have the coordinate* $(1, t)$*, taking* t *to infinity is well-defined in this quotient:* $(1, t) \sim (\frac{1}{t}, 1) \to (0, 1)$.

5.2 Rational Curves

Suppose we have a degree 1 curve C in $\mathbb{C}P^2$ parametrized by $\mathbb{C}P^1$, given as the zero locus of $a_0x_0 + a_1x_1 + a_2x_2$. Since a_i cannot all be zero, without loss of generality have $a_2 \neq 0$. Then we have the curves:

$$\phi : \mathbb{C}P^1 \to C \tag{32}$$

$$(x_0, x_1) \mapsto (x_0, x_1, -\frac{a_0x_0 + a_1x_1}{a_2}) \tag{33}$$

This is well-defined because the RHS is not (0,0,0) if $(x_0, x_1) \neq (0, 0)$, and so indeed defines a point of $\mathbb{C}P^2$. Furthermore, $\phi(\lambda(x_0, x_1)) = \lambda\phi(x_0, x_1)$ (it is equivariant under the quotient action).

More generally, let $f_0(x_0, x_1), ..., f_n(x_0, x_1)$ be any collection of homogeneous polynomials of the same degree d without common factors. Then the map

$$\mathbb{C}P^1 \to \mathbb{C}P^n \tag{34}$$

$$(x_0, x_1) \mapsto (f_0(x_0, x_1), ..., f_n(x_0, x_1)) \tag{35}$$

is well-defined, and its image (if f is nonconstant) is called a **rational curve**. It is called a rational curve because the restictions of such curves to the subset $U_0 \subset \mathbb{C}P^n$ are identified with curves parametrized by the rational functions $(f_1/f_0, ..., f_n/f_0)$.

For g_0, g_1 degree e polynomials, we can parametrize the curve in a new way with the degree de polynomial

$$(x_0, x_1) \mapsto (f_0(g_0(x_0, x_1), g_1(x_0, x_1)), ..., f_n(g_0(x_0, x_1), g(x_0, x_1))) \tag{36}$$

The degree of a rational curve is given by the minimal degree of a polynomial that parametrizes this curve.

Let $X \subset \mathbb{C}P^n$ be a hypersurface defined as the set of zeros of a homogeneous polynomial F. We say X is singular at $p \in X$ if $\frac{\partial F}{\partial x_i}(p) = 0 \forall i$.

5.3 Enumerative Geometry

The enumerative problem is the enumeration of rational curves of any given degree d contained in the hypersurface X, with X general. We can choose a parametrization and write such curves as $f : \mathbb{C}P^1 \to X$.

Let $X \subset \mathbb{C}P^4$ be a quintic threefold. "Quintic" because it is degree 5, and "threefold" because it has 3 free variables in $\mathbb{C}P^4$. We will describe a rational curve by

$$f(x_0, x_1) = (f_0(x_0, x_1), ..., f_4(x_0, x_1)) \tag{37}$$

and X by the zero-locus of the degree 5 homogeneous polynomial

$$F(x_0, ..., x_4) \tag{38}$$

The condition that the curve be contained in X is given by requiring that

$$F(f_0(x_0, x_1), ..., f_4(x_0, x_1)) = 0 \tag{39}$$

This means, for

$$f_j(x_0, x_1) = \sum_{i=0}^{d} a_{ij} x_0^i x_1^{d-i} \tag{40}$$

(homogeneous polynomials of degree d in (x_0, x_1) form a vector space of dimension $d+1$) we have

$$F = \sum_{k=0}^{5d} h_k(\{a_{ij}\}) x_0^k x_1^{5d-k} = 0 \tag{41}$$

which holds if and only if the $5d+1$ equations $h_k(\{a_{ij}\}) = 0$ hold. Note that there are $5(d+1)$ such a_{ij}s, so we expect a system of $5d+1$ euqations in $5(d+1)$ unknowns to have 4 free variables in its solution.

Because of this, it looks like we have infinitely many rational curves. But we can reparametrize with $f(g_0(x_0, x_1), g_1(x_0, x_1))$, where, to stay in degree d, g_0, g_1 are linear. Thus, with (g_0, g_1), we have a tuple of degree 1 polynomials. Degree 1 polynomials of x_0, x_1 form a vector space of dimension 2, so a tuple of these degree 1 polynomials form a 4-dimensional vector space. Thus we have a 4-4=0-dimensional space of solutions to $F(f_0, ..., f_n) = 0$, up to reparametrization.

Remark 2. *The Clemens Conjecture states that, for d a positive integer and $X \subset P^4$ a general quintic threefold, there are finitely many rational curves of degree d in X. This is known to be true for $d \leq 9$.*

Also, by allowing common factors in the f_i, we get infinitely any extraneous solutions.

Remark 3. *For $d = 1$, the number of curves (lines) is 2875. The number of degree 2 rational curves (conics) is 609250. The number of degree 3 rational curves is 317206375. Then physicists predicted the number of rational curves of all degrees using string theory. The worldsheet of a closed string is determined by the classical equations of motion to be an algebraic curve.*

5.4 Stable Maps

A **stable map** is a pseudoholomorphic (holomorphic with respect to its almost-complex structure J) map from a Riemann surface with at worst nodal singularities such that there are finitely many automorphisms that preserve its nodes and marked points.

Example 5. *Suppose that $C = \mathbb{C}P^1$ and $f : C \to \mathbb{C}P^n$ is a parametrized rational curve $(x_0, x_1) \mapsto (f_0(x_0, x_1), ..., f_n(x_0, x_1))$ of degree d. Let $g_0(x_0, x_1), g_1(x_0, x_1)$ be any pair of homogeneous linear polynomials without a common factor. Then the map*

$$g : \mathbb{C}P^1 \to \mathbb{C}P^1 \tag{42}$$

$$g(x_0, x_1) = (g_0(x_0, x_1), g_1(x_0, x_1)) \tag{43}$$

is an isomorphism from f to the stable map $(x_0, x_1) \mapsto (f_0(g_0(x_0, x_1), g_1(x_0, x_1)), ..., f_n(g_0(x_0, x_1), g_1(x_0, x_1)))$.

Thus a parametrized rational curve defines a stable map, but the isomorphism class of the stable map does not depend on the choice of parametrization. In particular, a rational curve determines a unique stable map, independent of the choice of parametrization.

The $n-$pointed **moduli space of stable maps** is the set of all isomorphism classes of degree d genus g stable maps to $\mathbb{C}P^n$, denoted $\overline{M}_{g,n}(\mathbb{C}P^n, d)$. This is actually a **stack**: It is a scheme mod a group action, the group being the automorphism group of parametrization. In the more abstract definition of a stack, the fibered category... category of groups...

6 Scary Mirror Symmetry

Homological mirror symmetry states that the Derived Category of Coherent Sheaves is the Fukaya category of its mirror, and the Derived Category of Coherent Sheaves of its mirror is the Fukaya category of the original manifold.