Symplectic Geometry & Topology Problems

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Problems assigned by Umut Varolgunes in Autumn 2019.

Question 1. *Find 5 different ways in which a function on $\mathbb{R}^{2n}$ can give rise to a canonical transformation of $T^*\mathbb{R}^n$.*

Proof. According to Weinstein, if we have a symplectomorphism from $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ to $T^*\mathbb{R}^{2n}$, we can identify canonical relations from $T^*\mathbb{R}^n$ to itself via functions on $T^*\mathbb{R}^{2n}$. We want to look at maps

$$g_i : \mathbb{R}^{2n} \to T^*\mathbb{R}^{2n} \tag{1}$$

$$g_i(x, y) \mapsto (x, y, \xi, \xi'), \text{ for } x, y \in \mathbb{R}^n \tag{2}$$

$$(x, y, \frac{\partial g_i}{\partial x}, \frac{\partial g_i}{\partial y}) \tag{3}$$

That correspond to the four canonical transformation functions given in the Wikipedia page:

$$\begin{aligned}
T^*\mathbb{R}^n \times T^*\mathbb{R}^n &\longrightarrow T^*\mathbb{R}^{2n} \\
(x, \xi) \times (y, \xi') &\longmapsto (x, y, \xi, \xi') \\
(x, \xi) \times (y, \xi') &\longmapsto (x, -\xi', \xi, y) \\
(x, \xi) \times (y, \xi') &\longmapsto (-\xi, y, x, \xi') \\
(x, \xi) \times (y, \xi') &\longmapsto (-\xi, -\xi', x, y)
\end{aligned}$$

From Weinstein, we know that such symplectomorphisms from $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ to $T^*\mathbb{R}^{2n}$ can be identified with certain canonical relations from $\mathbb{R}^n$ to $\mathbb{R}^n$. Since the four functions given on Wikipedia are in terms of the first two coordinates of $T^*\mathbb{R}^{2n}$, those are the coordinates in which our function is given. A fifth symplectomorphism is

$$T^*\mathbb{R}^n \times T^*\mathbb{R}^n \to T^*\mathbb{R}^{2n} \tag{4}$$

$$(x, \xi) \times (y, \xi) \mapsto (\frac{x-\xi}{2}, \frac{y-\xi'}{2}, \frac{x+\xi}{2}, \frac{y+\xi'}{2}) \tag{5}$$

This is a symplectomorphism because first it is obviously a bijection. Next, with our standard symplectic forms on $T^*\mathbb{R}^n$, $\sum_i dx_i \wedge d\xi_i$, give us $\sum_i (dx_i - d\xi_i) \wedge (dx_i + d\xi_i) = \sum_i dx_i \wedge d\xi_i - d\xi_i \wedge dx_i = \sum_i 2(dx_i \wedge d\xi_i)$. The coordinates with y, ξ' work the same way, giving us the standard symplectic form on $T^*\mathbb{R}^{2n}$ when dividing by 2. $\square$

Question 2. *Prove that* $GL(n, \mathbb{C}) \cap O(2n) = GL(n, \mathbb{C}) \cap Sp(n) = Sp(n) \cap O(2n) = U(n)$

Proof. First we prove $GL(n, \mathbb{C}) \cap O(2n) = U(n)$.

Lemma 1. *If an automorphism M on $\mathbb{R}^{2n}$ is compatible with $J : (p, q) \mapsto (-q, p), p, q \in \mathbb{R}^n$, then M is of the form*

$$M = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \tag{6}$$

Proof. We write J as

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

For an automorphism M to be compatible with J, we must have $J^T M J = M$. By inspection, $J^T = -J$, so $J^T M J = M \Rightarrow -JMJ = M \Rightarrow -J^2 M J = JM \Rightarrow MJ = JM$. Thus

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \tag{7}$$

$$\begin{pmatrix} Y & -X \\ W & -Z \end{pmatrix} = \begin{pmatrix} -Z & -W \\ X & Y \end{pmatrix} \Rightarrow \tag{8}$$

$$M = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \tag{9}$$

$\square$

Lemma 2. $GL(n, \mathbb{C}) \cap O(2n) = U(n)$

Proof. We can go from $GL(n, \mathbb{C})$ to elements of $GL(2n, \mathbb{R})$, when, if we write $M \in GL(n, \mathbb{C})$ with

$m_{ij} = x_{ij} + iy_{ij}$ as $M = X + iY$, for real matrices X, Y. Thus we have a map

$$m : \begin{pmatrix} M \end{pmatrix} \mapsto \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \tag{10}$$

Clearly m is injective, as the only way to get the identity in $GL(2n, \mathbb{R})$ is if $Y = 0, X = I_n$, which corresponds to the identity in $GL(n, \mathbb{C})$. We now consider elements of $GL(n, \mathbb{C})$ as the matrices constructed above. Taking $m(M^*)$, we get

$$\begin{pmatrix} M^* \end{pmatrix} = \begin{pmatrix} X - iY \end{pmatrix}^T = \begin{pmatrix} X^T - iY^T \end{pmatrix} \mapsto \begin{pmatrix} X^T & -Y^T \\ Y^T & X^T \end{pmatrix} = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}^T = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}^{-1} \tag{11}$$

where the last equality is due to us being in $O(2n)$. This is the inverse of $m(M)$, so $M \in U(n)$. $\square$

Lemma 3. $GL(n, \mathbb{C}) \cap Sp(n) = U(n)$

Proof. With this symplectic structure on $\mathbb{R}^{2n}$:

$$\Omega = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \tag{12}$$

Since we know that M is compatible with Ω, $\Omega = M^T \Omega M$. So,

$$\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}^T \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \Rightarrow \tag{13}$$

$$\begin{pmatrix} X^T & -Y^T \\ Y^T & X^T \end{pmatrix} \begin{pmatrix} Y & -X \\ X & Y \end{pmatrix} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \Rightarrow \tag{14}$$

$$\begin{pmatrix} X^T Y - Y^T X & -X^T X - Y^T Y \\ Y^T Y + X^T X & -Y^T X + X^T Y \end{pmatrix} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \tag{15}$$

This is possible if and only if $Y^T X - X^T Y = 0$ and $X^T X + Y^T Y = I_n$. Thus, $M^* M = (X - iY)^T (X + iY) = [X^T X + iX^T Y - iY^T X + Y^T Y] = [X^T X + Y^T Y + i(0)] = I_n$. Thus M is unitary. $\square$

Lemma 4. *$Sp(n) \cap O(2n) = U(n)$*

Proof. If a matrix M is in $Sp(n)$, then $M^T \Omega M = \Omega$. Furthermore, if $M \in O(2n)$, then $MM^T \Omega M = M\Omega \Rightarrow \Omega M = M\Omega$. For our Ω, we see that this is the same condition needed for Lemma 1, so thus

$$M = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \tag{16}$$

From Lemma 2, we see that such an M can be the image of some complex matrix $M' = X + iY$ under m. For M to be in $O(2n)$, we must have

$$\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}^T \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} = \begin{pmatrix} X^T & -Y^T \\ Y^T & X^T \end{pmatrix} \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \tag{17}$$

$$= \begin{pmatrix} X^T X + Y^T Y & X^T Y - Y^T X \\ Y^T X - X^T Y & Y^T Y + X^T X \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_{2n} \end{pmatrix} \tag{18}$$

where the last equality is true if and only if $X^T X + Y^T Y = I_n, Y^T X = X^T Y$. When we look at $M' := m^{-1}(M)$, we see that $(M')^* M' = (X - iY)^T (X + iY) = X^T X + iX^T Y - iY^T X + Y^T Y$. If X, Y are such that $X^T X + Y^T Y = I_n, Y^T X = X^T Y$, then this product is equal to I_n, so M' is unitary. □

□

Question 3. *The fundamental group of the Lagrangian Grassmannian $\Lambda(n)$ is free cyclic and its generator goes into the generator of the mapping induced by Det^2.*

Proof. Suppose we have a Lagrangian plane $\lambda \in \Lambda(n)$. This means that $\omega(v, w) = 0, \forall v, w \in \lambda$. Given another Lagrangian plane $\lambda' \in \Lambda(n)$, we know that an automorphism ϕ on $\mathbb{R}^{2n}$ from λ to λ' must preserve the symplectic structure as well as the Euclidean structure, as $(I\phi(v), \phi(w))$ must still be zero for λ' to be in $\Lambda(n)$. Thus ϕ must be in $O(2n) \cap Sp(n) = U(n)$ from above. We also have that $I\lambda$ is orthogonal to λ because $\omega(v, w) = (Iv, w) = 0, Iv \in I\lambda, \forall v, w \in \lambda$. Let ξ, ξ' be orthogonal framings of $\lambda, \lambda' \in \Lambda(n)$, respectively. Since we can have an automorphism from ξ to ξ', fix ξ. Since every ξ' can be written as $\phi(\xi)$ for some $\phi \in U(n)$, all ξ' are in the orbit of ξ, and so $U(n)$ acts transitively on $\Lambda(n)$. $O(n)$ preserves the subspace by simply rotating the

the real part of the frame ξ. However, this doesn't make the transformed plane transverse, Thus, $\Lambda(n) \cong U(n)/O(n)$, and we have a fibration

$$O(n) \to U(n) \to \Lambda(n) \tag{19}$$

Since, for $O \in O(n)$, $O^T O = I_n, \det(O^T)\det(O) = 1$, so $\det(O) = \pm 1$. The square of the determinant of some $\phi \in U(n)$ carrying the plane where the first n entries are 0 into λ depends only on λ. Thus we have the map

$$\overset{2}{\det} : \Lambda(n) \to S^1 \tag{20}$$

Denote $S\Lambda(n)$ by the set $\{\lambda \in \Lambda(n) | \det^2 \lambda = 1\}$. In the same way as above, $S\Lambda(n) \cong SU(n)/SO(n)$ is a submanifold of $\Lambda(n)$. Finally, we define a map

$$z^2 : S^1 \to S^1 \tag{21}$$

$$e^{i\varphi} \mapsto e^{2i\varphi} \tag{22}$$

Thus we have a commutative diagram of six fibrations:

$$\begin{array}{ccccc}
SO(n) & \longrightarrow & O(n) & \xrightarrow{\det} & S^0 \\
\downarrow & & \downarrow & & \downarrow \\
SU(n) & \longrightarrow & U(n) & \xrightarrow{\det} & S^1 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle z^2} \\
S\Lambda(n) & \longrightarrow & \Lambda(n) & \xrightarrow{\det^2} & S^1
\end{array}$$

With these fibrations, we have the long exact sequences of homotopy groups:

$$\begin{array}{ccccccccc}
\pi_2(S^0) & \longrightarrow & \pi_1(SO(n)) & \longrightarrow & \pi_1(O(n)) & \xrightarrow{\det^*} & \pi_1(S^0) & \longrightarrow & \pi_0(SO(n)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\pi_2(S^1) & \longrightarrow & \pi_1(SU(n)) & \longrightarrow & \pi_1(U(n)) & \xrightarrow{\det^*} & \pi_1(S^1) & \longrightarrow & \pi_0(SU(n)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle z^{2*}} & & \\
\pi_2(S^1) & \longrightarrow & \pi_1(S\Lambda(n)) & \longrightarrow & \pi_1(\Lambda(n)) & \xrightarrow{\det^{2*}} & \pi_1(S^1) & \longrightarrow & \pi_0(SU(n)) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \pi_0(SO(n)) & \longrightarrow & \pi_0(\Lambda(n)) & \longrightarrow & \pi_0(S^0) & &
\end{array}$$

First we examine $\pi_1(SU(n))$. This is the complex rotation group for unit vectors in $\mathbb{C}^n \cong \mathbb{R}^{2n}$, or S^{2n-1}. We look at the stabilizer subgroup for a point in S^{2n-1}. By spherical symmetry, consider

$(1, 0, ..., 0) \in S^{2n-1}$. A stabilizing subgroup of $SU(n)$ for this point is $SU(n-1)$:

$$SU(n-1) \to SU(n) \tag{23}$$

$$A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \tag{24}$$

Due to this embedding, we have a fibration

$$SU(n-1) \to SU(n) \to S^{2n-1} \tag{25}$$

$$\pi_1(SU(n-1)) \to \pi_1(SU(n)) \to \pi_1(S^{2n-1}) \tag{26}$$

First we calculate $\pi_1(SU(1)) \cong 0$, since $SU(1)$ is a singleton. Inductively, we have

$$\pi_1(SU(n-1)) \to \pi_1(SU(n)) \to \pi_1(S^{2n-1}) \Rightarrow \tag{27}$$

$$0 \to \pi_1(SU(n)) \to 0 \tag{28}$$

Therefore, $\pi_1(SU(n)) \cong 0, \forall n$. Next we prove that $SO(n)$ is path-connected. By the same logic as above, $S^n \cong SO(n+1)/SO(n)$. Suppose we have a path $I : [0, 1] \to S^n$. Lift this path to $SO(n+1)$ by the inverse quotient. If this path breaks, the remaining points of the path are contained in $SO(n)$. Inductively, since $SO(1)$ is a singleton, it is path-connected, so $SO(n)$ is path connected, and so $\pi_0(SO(n)) \cong 0$. Our exact sequence diagram is then

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(SO(n)) \arrow[r] \arrow[d] & \pi_1(O(n)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \pi_1(U(n)) \arrow[r] \arrow[d] & \mathbb{Z} \\
0 \arrow[r] & \pi_1(S\Lambda(n)) \arrow[r] \arrow[d] & \pi_1(\Lambda(n)) \arrow[r] & \mathbb{Z} \\
& 0 & &
\end{tikzcd}$$

Due to exactness, $\pi_1(S\Lambda(n)) \cong 0$, and therefore $S\Lambda(n)$ is simply connected, and therefore path-connected, so $\pi_0(S\Lambda(n)) \cong 0$. Therefore, our exact sequence is

$$\pi_1(S\Lambda(n)) \to \pi_1(\Lambda(n)) \to \mathbb{Z} \to \pi_0(S\Lambda(n)) \tag{29}$$

$$0 \to \pi_1(\Lambda(n)) \to \mathbb{Z} \to 0 \tag{30}$$

Hence $\pi_1(\Lambda(n)) \cong \mathbb{Z}$. □

Question 4. *Work out moment maps for the Hamiltonian actions of* $\mathbb{R}^3$ *and* $SO(3)$ *on* $T^*\mathbb{R}^3$*, corresponding to translations and rotations of* $\mathbb{R}^3$*. Make the connection with linear and angular momentum from classical mechanics.*

Proof. We check out Lie group actions of $\mathbb{R}^3$ and $SO(3)$ on $T^*\mathbb{R}^3$:

Let $G = \mathbb{R}^3$. Then $\mathfrak{g} \cong \mathfrak{g}^* \cong T_0\mathbb{R}^3 \cong \mathbb{R}^3$. We identify $T^*\mathbb{R}^3 \cong \mathbb{R}^6 := \{(q,p) \in \mathbb{R}^3 \times \mathbb{R}^3\}$, with standard symplectic structure $\omega = \sum_i^3 dq_i \wedge dp_i$. We have $\mathbb{R}^3$ act on $T^*\mathbb{R}^3$ by the map

$$\psi : \mathbb{R}^3 \to \mathrm{Sym}(T^*\mathbb{R}^3, \omega) \tag{31}$$

$$\psi(a) \mapsto [(q,p) \to (q+a,p)] \tag{32}$$

We know that $\psi(a)$ is a symplectic action, because $\psi(a)^*\omega(\dot{q},\dot{q}') =$

$\omega(d\psi(a)(\dot{q}), d\psi(a)(\dot{q}'))$, and $d(q+a) = dq + da = dq = dId$. Inspired by classical mechanics, we propose that our momentum map is given by

$$\mu : T^*\mathbb{R}^3 \to \mathfrak{g}^* \tag{33}$$

$$\mu(q,p) \mapsto p \tag{34}$$

We check that this is a moment map. First, we calculate the map μ^a:

$$\mu^a((q,p)) = \langle \mu((q,p)), a\rangle = \langle p, a\rangle = p(a) = a_1p_1 + a_2p_2 + a_3p_3 \tag{35}$$

$$d\mu^a = a_1dp_1 + a_2dp_2 + a_3p_3 \tag{36}$$

with $a^\#$ the vector field generated by the one-parameter subgroup $\{\exp(ta) | t \in \mathbb{R}\} \subset \mathbb{R}^3$, given by

$a_1p_1 + a_2p_2 + a_3p_3$. We examine $\iota_{a^\#}\omega$. With the standard symplectic form, this is equal to

$$\begin{pmatrix} 0 & 0 & 0 & a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} 0 & -Id_3 \\ Id_3 & 0 \end{pmatrix} = a_1dp_1 + a_2dp_2 + a_3dp_3 \tag{37}$$

Thus, μ^a is a Hamiltonian function for the covector $a^\# = a_1p_1 + a_2p_2 + a_3p_3$. We can also see that $\mu \circ \psi(a) = \mu$, because $\mu((q, p)) = \mu((q + a, p)) = p$. Furthermore, the coadjoint action is the identity:

$$\mathrm{Ad}_g^* p(\dot{q}) = \langle p, g\dot{q}g^{-1}\rangle = \langle p, \dot{q}\rangle = p(\dot{q}) \tag{38}$$

because the action by translation acts by identity on the tangent vectors of $\mathbb{R}^3$. Thus, $\mu \circ \psi(a) = \mathrm{Ad}_g^* \circ \mu = \mu$, so μ is a moment map.

Now let $G = SO(3)$. Let $SO(3)$ act on $T^*\mathbb{R}^3$ by the map

$$\psi : SO(3) \to \mathrm{Sym}(T^*\mathbb{R}^3, \omega) \tag{39}$$

$$\psi(R) \mapsto [(q, p) \to (Rq, Rp)] \tag{40}$$

This is a symplectic action because the usual symplectic product is

$$\omega(\dot{q}, \dot{q}') = (J\dot{q}, \dot{q}') = (J(q, v_q), (q', v_{q'})) = ((-v_q, q), (q', v_{q'})) = -v_q^T q' + q^T v_{q'} \Rightarrow \tag{41}$$

$$\omega(\psi(R)(\dot{q}), \psi(R)(\dot{q}')) = (J(Rq, Rv_q), (Rq', Rv_{q'})) = ((-Rv_q, Rq), (Rq', Rv_{q'})) \tag{42}$$

$$= -v_q^T R^T Rq + (q')^T R^T Rv_{q'} \tag{43}$$

$$= -v_q^T q' + q^T v_{q'} \tag{44}$$

In order to find $\mathfrak{so}(3)$, we find conditions on A such that $e^A \in SO(3)$. $e^{A^T} = (e^A)^T$, so we require that $(e^A)^T e^A = e^{A^T} e^A = e^{A^T + A} = Id$. Thus we require that A is skew-symmetric. By Jacobi's formula, $\det(e^A) = e^{\mathrm{Tr}(A)}$. Since we require $\det(e^A) = 1, \mathrm{Tr}(A)$ must be zero. This is already satisfied if A is skew-symmetric, so, since this exponential map is surjective, $\mathfrak{so}(3)$ is the set of skew-symmetric matrices. All of these matrices are of the form

$$\begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix} \tag{45}$$

Thus we identify $\mathbb{R}^3$ with $\mathfrak{so}(3)$ via the isomorphism

$$\phi : (\mathbb{R}^3, [\cdot,\cdot] = \times) \to (\mathfrak{so}(3), [\cdot,\cdot] = \text{usual commutator}) \tag{46}$$

$$\phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \tag{47}$$

$$:= x \mapsto A \tag{48}$$

The matrix defined above gives us a way to take the cross product between two vectors: $A\xi = x \times \xi$. Again inspired by classical mechanics, we propose that our momentum map is given by

$$\mu : T^*\mathbb{R}^3 \to \mathfrak{g}^* \tag{49}$$

$$\mu((q,p)) \mapsto q \times p \tag{50}$$

We then have

$$\mu^a((q,p)) = \langle \mu((q,p)), a \rangle = \langle q \times p, a \rangle = (q \times p) \cdot a \tag{51}$$

First we examine $d\mu^{(1,0,0)}((q,p))$. This is equal to $d(q_2p_3 - q_3p_2) = p_3dq_2 + q_2dp_3 - p_2dq_3 - q_3dp_2$. Now we check that this is equal to $\iota_{t(1,0,0)}\omega$. Under our correspondence,

$$(1,0,0) \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{52}$$

Thus $\iota_{t(1,0,0)}\omega$ is equal to

$$t\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & -Id_3\\ Id_3 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\end{pmatrix} := M \tag{53}$$

Then we have

$$(q_1, q_2, q_3, p_1, p_2, p_3)M = (0, p_3, -p_2, 0, -q_3, q_2) = p_3 dq_2 - p_2 dq_3 - q_3 dp_2 + q_2 dp_3 \tag{54}$$

This is the same as our expression for $d\mu^{(1,0,0)}$. The same works for the other generators $(0,1,0)$ and $(0,0,1)$. Next we check that $\mu \circ \psi(R) = \mathrm{Ad}^*_R \circ \mu$:

$$\mu((Rq, Rp)) = \mathrm{Ad}^*_R(q \times p) \tag{55}$$

$$(Rq \times Rp)(X) = \langle q \times p, \mathrm{Ad}_{R^{-1}} X\rangle \tag{56}$$

$$R(q \times p)(X) = \langle q \times p, R^{-1}XR\rangle \tag{57}$$

$$R(q \times p)(X) = \sum_i (q \times p)_i \tag{58}$$

$$= \langle q \times p, (R^{-1}X)_\times\rangle \tag{59}$$

where the $\times$ subscript denotes mapping the skew-symmetric matrix $R^{-1}XR$ into the $\mathbb{R}^3$ lie algebra. It remains to prove that $\langle Rq, p\rangle = \langle q, R^{-1}p\rangle$. $(R^{-1}p)_i = \sum_j R^{-1}_{ij} p_j$, and $\langle q, R^{-1}p\rangle = \sum_{i,j} q_i R^{-1}_{ij} p_j = R^{-1}_{ji} q_i p_j = \langle (R^{-1})^T q, p\rangle$, and since $R \in SO(3)$, this is equal to $\langle Rq, p\rangle$. Thus,

$$\langle q \times p, (R^{-1}X)_\times\rangle = R(q \times p)(X) \tag{60}$$

Thus $\mu((q, p)) = q \times p$ is a moment map.

Connecting to classical mechanics, p is linear momentum and $q \times p$ is angular momentum. □

Question 5. *Let $\lambda_1, ..., \lambda_n$ be positive real numbers and $X = S^2_{\lambda_1} \times ... \times S^2_{\lambda_n}$, where S^2_λ denotes the unit two sphere with invariant area form re-scaled by λ. The group $K = SO(3)$ acts diagonally on $X = (S^2)^n$ with moment map*

$$\Phi : X \to \mathfrak{t}^\vee \cong \mathbb{R}^3, (x_1, ..., x_n) \mapsto x_1 + ... + x_n \tag{61}$$

The symplectic quotient is the moduli space of closed n-gons with lengths $\lambda_1, ..., \lambda_n$

$$X//SO(3) = \{(x_1, ..., x_n) \in (\mathbb{R}^3)^n | \; ||x_j||\lambda_j, x_1 + ... + x_n = 0\}/SO(3) \tag{62}$$

Its topology depends on the choice of $\lambda_1, ..., \lambda_n$, see for example Hausmann-Knutson "The cohomology ring of polygon spaces". In general there are a finite number of "chambers" in which the topology of $X//SO(3)$ is constant. The chambers in which $X//SO(3)$ is non-empty are described by the following:

Lemma 5. *$X//SO(3) = \emptyset$ if and only if $\lambda_j \leq \sum_{i \neq j} \lambda_i$ for all $j = 1, ..., n$.*

Proof. In Example 3.3.3, the manifold is the product of 2-spheres with a new symplectic form: the n^{th} sphere is the symplectic manifold $(S^2, \lambda_n \omega) =: S^2_{\lambda_n}$, so the manifold X is $S^2_{\lambda_1} \times ... \times S^2_{\lambda_n}$. The regular symplectic form ω on S^2 is found by integrating in the normal direction of the volume form in $\mathbb{R}^3$ under the inclusion $S^2 \to \mathbb{R}^3$: $\omega = \iota_{x\dot{x}+y\dot{y}+z\dot{z}}(dx \wedge dy \wedge dz) = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$ at each point. The Lie group $SO(3)$ has its Lie algebra $\mathfrak{so}(3)$ isomorphic to $\mathbb{R}^3$ as shown in the previous problem. That $SO(3)$ "acts diagonally" just means that it acts on each factor $S^2_{\lambda_n}$ of X. The momentum map is given by

$$\mu : X \to \mathfrak{so}(3) \tag{63}$$

$$X \to \mathbb{R}^3 \cong \mathfrak{so}(3) \tag{64}$$

$$(x_1, ..., x_n) \mapsto \phi(x_1 + ... + x_n) \tag{65}$$

This is a moment map because $(M_1, \omega_1, \mu_1) \times (M_2, \omega_2, \mu_2)$ has moment map given by $\pi_1^* \mu_1 \times \pi_2^* \mu_2$, where π_1 is the projection map onto the i^{th} factor. This uses the diagonal map of the lie algebra $\mathfrak{g} \mapsto \mathfrak{g} \times \mathfrak{g}$ and dual $\mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^*$ given by $(\xi_1, \xi_2) \mapsto \xi_1 + \xi_2$. Knowing this, we can just examine the momentum map for each $S^2_{\lambda_n}$. For the normal sphere, the action on $SO(3)$ is rotation about two axes while fixing the third by symmetry. By the same logic as above, $\omega_x(u,v) = \langle x, u \times v\rangle$, so the inclusion μ of S^2 into $\mathbb{R}^3 \cong \mathfrak{so}(3)$ satisfies $\iota_{a^\#}\omega = \langle \mu, a\rangle$. We can also see that μ is equivariant. By symmetry, fix a point under the rotation $R \in SO(3)$ so that $\mu(x) \circ \phi(R)(Y) = \mu(x)(Y) = x(Y)$. Then $\mathrm{Ad}^*_R x = \langle x, \mathrm{Ad}_{R^{-1}} Y\rangle = \langle x, R^{-1} Y R\rangle = \langle x, R^{-1} \cdot Y\rangle = \langle Rx, Y\rangle = x(Y)$. Thus this is a moment map. Since $X//SO(3) := \mu^{-1}(0)/SO(3)$, we examine the inverse image of $\mu(0)$. 0 here corresponds to the matrix of all zeros in $\mathfrak{so}(3)$. Under our isomorphism to $\mathbb{R}^3$, this corresponds to the zero vector. Thus $\mu^{-1}(\mathbf{0})$ is the set of $x_1 + ... + x_n = \mathbf{0}$. To ensure that these are vectors in X, we require that the j^{th} vector x_j is an element of the the 2-sphere scaled by λ_j. Thus $X//SO(3) = \{(x_1, ..., x_n) \in (\mathbb{R}^3)^n | ||x_j|| = \lambda_j, x_1 + ... + x_n = \mathbf{0}\}$

For $X//SO(3)$ to be nonempty we need $\lambda_j \leq \sum_{i \neq j} \lambda_i$. For $n = 2$, $x_1 = -x_2$, so $||x_1|| = ||x_2||$ which is satisfied. For $n = 3$, this creates a triangle if $x_1 + x_2 + x_3 = 0$ (each x_i can be thought of as a side of the polygon). Thus this is equivalent to the triangle inequalities: $||x_1|| \leq ||x_2|| + ||x_3||, ||x_2|| \leq ||x_1|| + ||x_3||, ||x_3|| \leq ||x_2|| + ||x_1||$. For $n > 3$, assume without loss of generality that $\lambda_1 \geq ... \geq \lambda_n$. We know that there exists a $j \leq n-1$ such that $\lambda_1 < |\lambda_2 + ... + \lambda_j - \lambda_{j+1} - ... - \lambda_n|$. To see this, suppose not. Then there exists a $j \leq n-1$ such that $\lambda_2 + ... + \lambda_j - \lambda_{j+1} - ... - \lambda_n$ is positive and that $\lambda_2 + ... + \lambda_{j-1} - \lambda_j - ... - \lambda_n$ is negative. The only case where this is not true is if $\lambda_2 \geq \lambda_3 + ... + \lambda_n$. If this is true, then $\lambda_1 < |\lambda_2 - ... - \lambda_n|$ implies that $\lambda_1 < \lambda_2$, a contradiction. Suppose

$$\lambda_1 < |\lambda_2 + ... + \lambda_j - \lambda_{j+1} - ... - \lambda_n|, \tag{66}$$

$$\lambda_1 < |\lambda_2 + ... + \lambda_{j-1} - \lambda_j - ... - \lambda_n|, \text{ where} \tag{67}$$

$$\lambda_2 + ... + \lambda_j - \lambda_{j+1} - ... - \lambda_n > 0, \tag{68}$$

$$\lambda_2 + ... + \lambda_{j-1} - \lambda_j - ... - \lambda_n < 0 \tag{69}$$

This implies that $\lambda_j > \lambda_1$, a contradiction. Thus, there exists a (relabeled) j where $\lambda_1 > |\lambda_2 + ... + \lambda_j - \lambda_{j+1} - ... - \lambda_n|$. Now treat $\lambda_1, \lambda_2 + ... + \lambda_j$, and $\lambda_{j+1} + ... + \lambda_n$ as the sides of a triangle, and the condition follows. Then $x_1 + ... + x_n = 0$ with their respective required norms if and only if $\lambda_i \leq \sum_{i \neq j} \lambda_j$. □

Question 6. *Prove the exact sequence*

$$\tilde{H} \to \tilde{G} \xrightarrow{F} H^1(M;\mathbb{R}) \tag{70}$$

what is the corresponding exact sequence of Lie algebras?

Proof. The exact sequence is

$$0 \to \tilde{H} \to \tilde{G} \xrightarrow{F} H^1(M,\mathbb{R}) \to 0 \tag{71}$$

where $\tilde{G}$ is the universal cover of the group G of the path-connnected component of the group of symplectomorphisms containing the identity, $\tilde{H}$ is the subgroup of homotopy classes of paths in G with fixed endpoints to a Hamiltonian path, M is a compact symplectic manifold without boundary, and $F(\{\phi_t\}) = \int_0^1 \lambda_t dt$, for $\lambda_t(\cdot) = \omega(\frac{d\phi_t}{dt}, \cdot)$ for all $t \in [0,1]$. The inclusion of the subgroup $\tilde{H}$ into $\tilde{G}$ is indeed injective; if another element of $\tilde{H}$ were mapped to the identity of $\tilde{G}$, the image would cease to be isomorphic to the subgroup and this would not be the inclusion. In addition, F is surjective; take a closed 1-form u. u gives a symplectic vector field $X_t = X$ on M that is constant in time. Let ϕ_t be the family of symplectomorphisms generated by X_t. Then

$$F(\{\phi_t\}) = \int_0^1 [\omega(X_t,\cdot)]dt = \int_0^1 [\omega(X,\cdot)]dt = [\omega(X,\cdot)] = [u] \tag{72}$$

Thus we can create a map $s : H^1(M;\mathbb{R}) \to \tilde{G}$ such that $F \circ s = Id$, given by $s(u) = \exp(tX)$, where X is the vector canonically identified with u (X is such that $\iota_X\omega = u$). Now we prove that the image of the inclusion of paths homotopic with fixed endpoints to Hamiltonian paths have flux image 0. If ϕ is Hamiltonian, it is the endpoint of a Hamiltonian isotopy ϕ_t. This isotopy corresponds to a family of Hamiltonian functions H_t. Then

$$F(\{\phi_t\}) = \int_0^1 \omega(\frac{d\phi_t}{dt},\cdot)dt = \int_0^1 \iota_{\frac{d\phi_t}{dt}}\omega dt = \int_0^1 dH_t dt = 0 \tag{73}$$

Suppose $F(\{\phi_t\}) = 0$. Then there exists a function $f : M \to \mathbb{R}$ such that $\int_0^1 (\frac{d\phi_t}{dt},\cdot)dt = df$. Consider the Hamiltonian flow ϕ_f^s of f. Now it suffices to prove this claim for $(\phi_f^s)^{-1} \circ \phi$. Call this new path $\psi_t := \phi_{2t}$ for $0 \le t \le \frac{1}{2}$ and $\phi_f^{1-2t} \circ \phi_1$ for $\frac{1}{2} \le t \le 1$. ψ_t is generated by a smooth family of vector fields X_t such that $\int_0^1 X_t dt = 0$. Therefore we consider ϕ as some ϕ_t for some isotopy with $\int_0^1 X_t dt = 0$. Define a vector field $Y_t := \int_0^1 X_s ds$, and let θ_t^p be the flow generated by Y_t. Then $\frac{d\theta_t^p}{dp} = Y_t \circ \theta_t^p$. From before we know that $Y_1 = 0$, and Y_0 is trivially 0. Then $\theta_0^p = \theta_1^p = Id$ for all p. Call $\psi_t := \theta_t^1 \circ \phi_t$. This is an isotopy from Id to ϕ. Then

$$F(\{\psi_t\}) = F(\{\theta_t^1\}) + F(\{\phi_t\}) \text{ if F is a homomorphism} \tag{74}$$

$$= \int_0^1 [\omega(Y_t, \cdot)]ds + \int_0^1 [\omega(X_t, \omega)]dt \text{ if F is homotopy invariant} \tag{75}$$

$$= [\omega(Y_t, \cdot)] + \int_0^1 [\omega(X_t, \omega)]dt \text{ by integrating} \tag{76}$$

$$= -\int_0^1 [\omega(X_t, \cdot)]dt + \int_0^1 [\omega(X_t, \cdot)]dt = 0 \tag{77}$$

Thus, if F is indeed homotopy invariant and a homomorphism, ψ_t is a Hamiltonian isotopy from Id to ϕ, and thus is the image of the inclusion from $\tilde{H}$. In proving that F is a homomorphism, we examine the image of F of a juxtaposition of paths. Let χ_t be defined as ϕ_{2t} for $0 \leq t \leq \frac{1}{2}$ and $\psi_{2t-1} \circ \phi_1$ for $\frac{1}{2} \leq t \leq 1$.

$$F(\{\chi_t\}) = \int_0^1 \omega(\frac{d}{dt}\chi_t, \cdot)dt \tag{78}$$

$$= \int_0^{1/2} \omega(\frac{d\phi_{2t}}{dt}, \cdot)dt + \int_{1/2}^1 \omega(\frac{d\psi_{2t-1}}{dt}, \cdot)dt \tag{79}$$

$$= \int_0^1 \omega(\frac{d\phi_t}{dt}, \cdot)dt + \int_0^1 \omega(\frac{d\psi_t}{dt}, \cdot)dt \tag{80}$$

$$= F(\{\phi\}) + F(\{\psi\}) \tag{81}$$

Checking that F's homotopy invariance, we use the association of 1^{st} de Rham cohomology to the fundamental group, which leads us toward homotopy invariance:

$$H^1(M;) \leftarrow \mathrm{Hom}(\pi_1(M); \mathbb{R}) \tag{82}$$

$$\pi_1(M) \to \mathbb{R} \tag{83}$$

$$\gamma(s) \in \mathbb{R}/\mathbb{Z} \mapsto \int_0^1 \int_0^1 \omega(\frac{d\phi_t}{dt}(\gamma(s)), \frac{d\gamma}{ds}(s))dtds \tag{84}$$

Since the symplectic form is preserved and thus stays closed, the integral depends only on the homotopy class of γ. Define $\beta(s,t) := \phi_t^{-1}(\gamma(s))$, so $\phi_t(\beta) = \gamma \Rightarrow d\phi_t(\beta)\frac{\partial \beta}{\partial s} = \frac{d\gamma}{ds}(s), d\phi_t(\beta)\frac{\partial \beta}{\partial t} =$

$-\frac{d\phi_t}{dt}(\gamma(s))$, so the integral becomes

$$\int_0^1 \int_0^1 \omega(-\frac{\partial \beta}{\partial t}, \frac{\partial \beta}{\partial s}) dsdt = \int_0^1 \int_0^1 \omega(\frac{\partial \beta}{\partial s}, \frac{\partial \beta}{\partial t}) dsdt = \int_{\mathbb{R}/\mathbb{Z}} \int_0^1 \beta^* \omega \tag{85}$$

This integral only depends on the homotopy class of β, given that $\beta(s+1,t) = \beta(s,t), \beta(s,1) = \phi_1^{-1}(\beta(s,0))$. Thus the integral only depends on the homotopy class of ϕ_t with $\phi_0 = Id, \phi_1 = \phi$.

The corresponding exact sequence of Lie algebras is

$$0 \to \mathbb{R} \xrightarrow{a} C^\infty(M) \xrightarrow{b} \chi(M,\omega) \xrightarrow{c} H_1(M;\mathbb{R}) \to 0 \tag{86}$$

where a maps to real-valued smooth functions on M, b maps Hamiltonian functions H to the vectors X_H, and b maps vectors X to the class $[\omega(X,\cdot)]$.

The classical flux conjecture is explicitly stated as follows: the subgroup in $H^1(M;\mathbb{R})$ defined by $\pi_1(G) \xrightarrow{F} H^1(M;\mathbb{R})$ is a discrete subgroup for all symplectic manifolds. Unpacking this definition, we take loops with basepoint the identity in G and try to deduce properties even for those that wander outside of the identity's Weinstein neighborhood, where inside a Hamiltonian isotopy is known. That $F(\pi_1(G))$ is a discrete subgroup of H^1 is equivalent to H, the group of Hamiltonian symplectomorphisms being a submanifold of the group of symplectomorphisms. The flux conjecture was proved by Kaoru Ono in 2006. □